Form Factors, Grey Bodies and Radiation Conductances (Radks)

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Introduction

With today's analysis tools, large, complex thermal radiation problems are easily solved;

But, as with any analytical tool, lack of an understanding of the fundamental equations and technique limitations may leave you with the wrong answer;

Whether you are a new engineer or a seasoned veteran, an understanding of the techniques employed by these powerful analysis tools is crucial.

Overview

This lesson explores thermal radiation analysis techniques for form factors, grey body factors and Radks.

Scope of this Lesson

The Radk and its role in thermal analysis;

Form factor calculation techniques;

Grey body factor calculation techniques;

Radiation Conductance (Radk).

Lesson Roadmap

Fundamentals

What is a Radk?

Form Factors

Exact Solutions

Double Area Summation

Nusselt Sphere

Crossed-String Method

The "World's Worst Dart Player"

Form Factors Using Monte Carlo Ray Tracing

Statistical Error

Form Factor Reciprocity and Combinations

Grey Body Factors

Approximating $\mathscr{F}$

Gebhart's Method

Imperfect Form Factor Sums

Calculating Grey Body Factors Using Monte Carlo Ray Tracing

The Fence Problem

Turning the Grey Body Factor into a Radk

Application in Analysis

Fundamentals

The Blackbody

A blackbody is the perfect absorber and emitter of radiant energy;

The Stefan-Boltzmann law shows that energy radiated from a blackbody is a function only of its absolute temperature, T:

$$\dot{q} = \sigma T^4$$

where $\sigma = 5.67 \times 10^{-8}$ W/m^2 K^4.

The Grey Body

Most objects are not perfect blackbody absorbers or emitters -- they are said to be "grey";

To account for imperfect absorption and emission, the Stefan-Boltzmann equation is scaled by an proportionality term, ε;

$$\dot{q} = \varepsilon \sigma T^4$$

where ε is a value between 0 and unity.

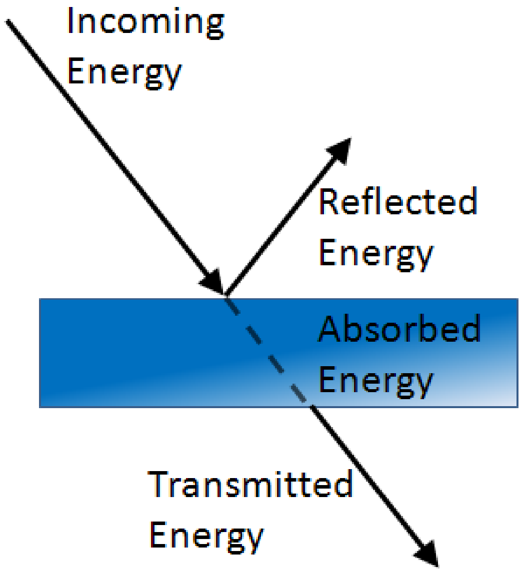
Conservation of Energy

Conservation of energy tells us that the fraction of energy absorbed by a surface, α_λ, plus the energy reflected, ρ_λ, plus the energy transmitted, τ_λ, must equal the energy incident on that unity:

$$\alpha_\lambda + \rho_\lambda + \tau_\lambda = 1$$

For an opaque surface, $\tau_\lambda = 0$ and the equation simplifies to:

$$\alpha_\lambda + \rho_\lambda = 1$$



Conservation of Energy

If we limit our discussion of absorption and emission to the same portion of the spectrum (or, more precisely, to a given wavelength, λ), we can say:

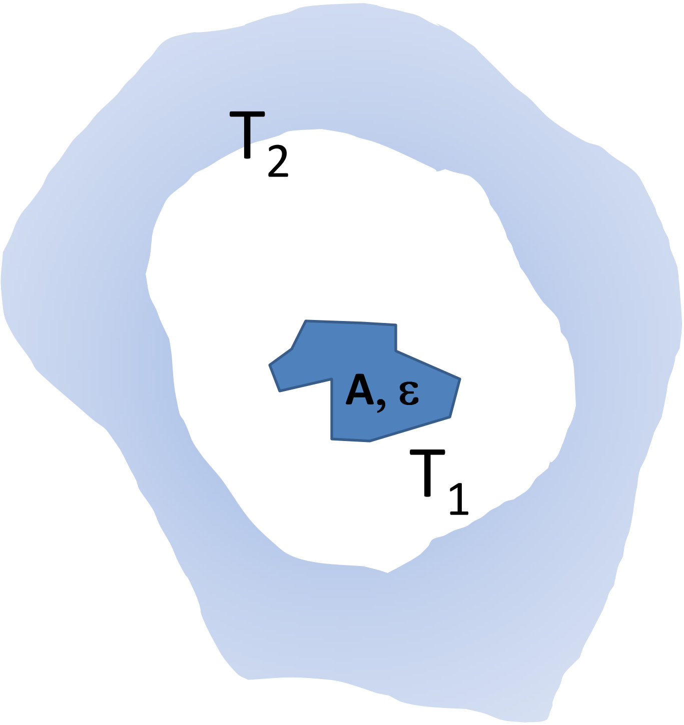
$$\alpha_\lambda = \varepsilon_\lambda$$

For the purpose of our discussion, *we're going to use* ε *for, both, absorption and emission in the infrared spectrum.*

Kirchhoff's Law

This says, for a given wavelength, λ, an object's ability to absorb radiant energy is equal to its ability to radiate energy;

Consider an object with surface area, A, and emittance, ε, at temperature, T_1, introduced into an oven at T_2.

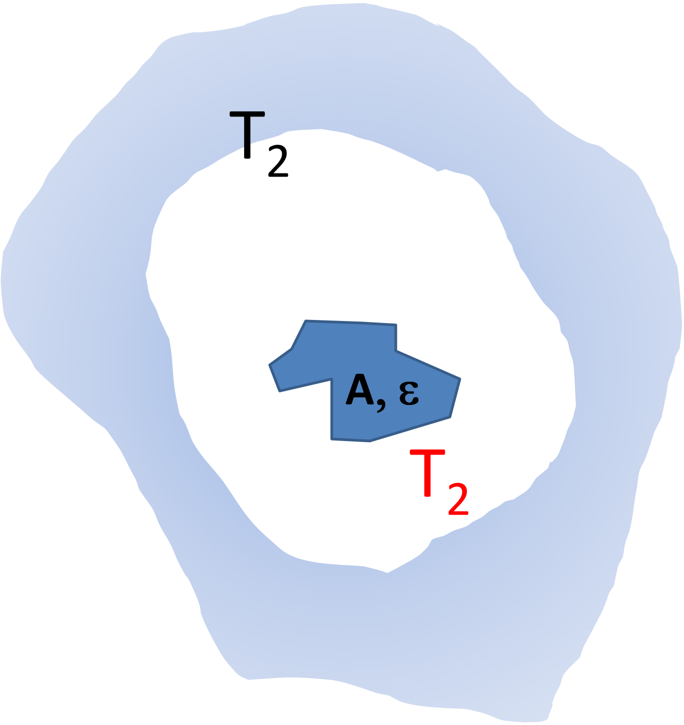


Kirchhoff's Law

At steady state, there is no net heat transfer so the object warms until is reaches T_2;

When the object reaches T_2, the amount of energy per unit time being absorbed is equal to that being radiated.

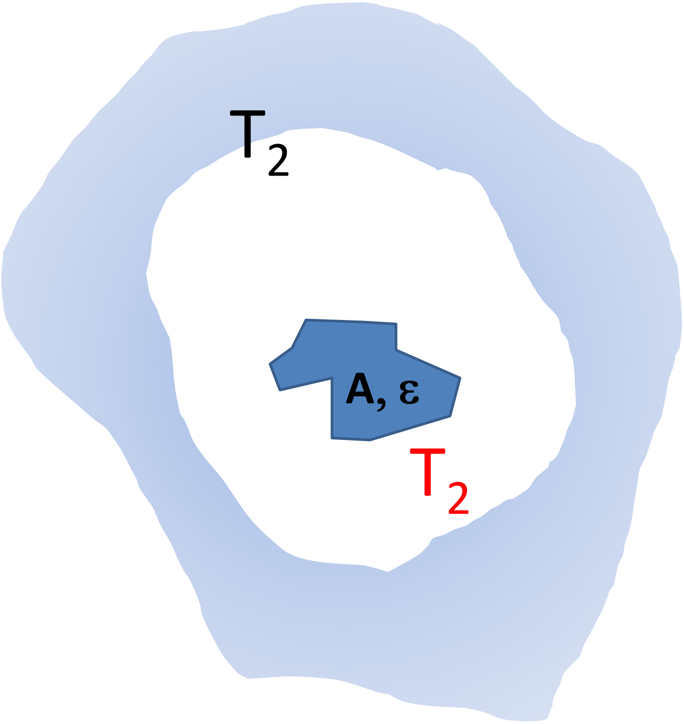
$$\alpha\sigma A T_2^4 = \varepsilon\sigma A T_2^4$$



Kirchhoff's Law

The only way for this to be true is for:

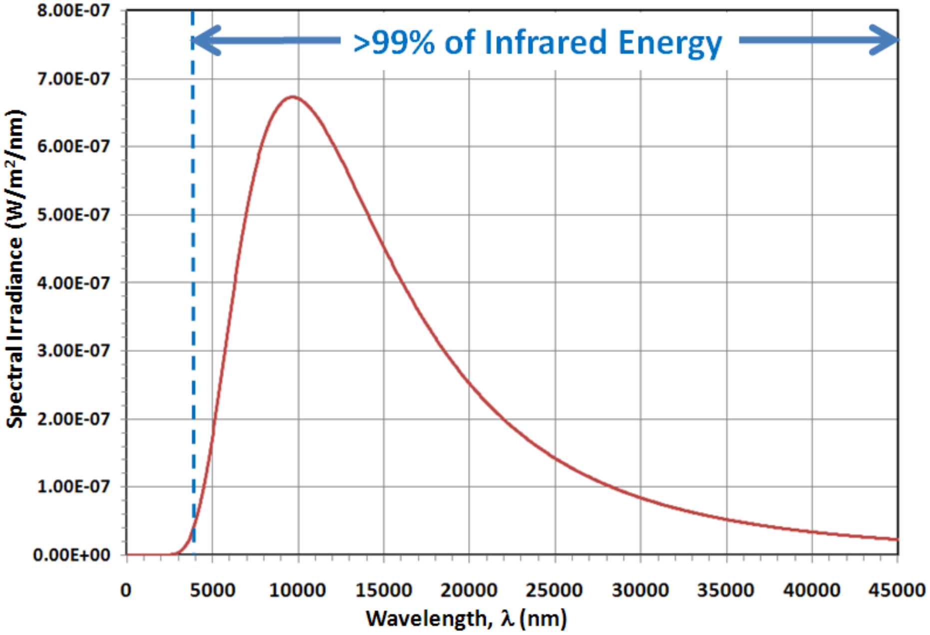
$$\varepsilon = \alpha$$



Emittance

For thermal radiation, we generally consider the range of $\lambda > 4000$ nm to be in the infrared range (Ref. 1);

We use ε to describe absorption and emission of radiation in this wavelength range.



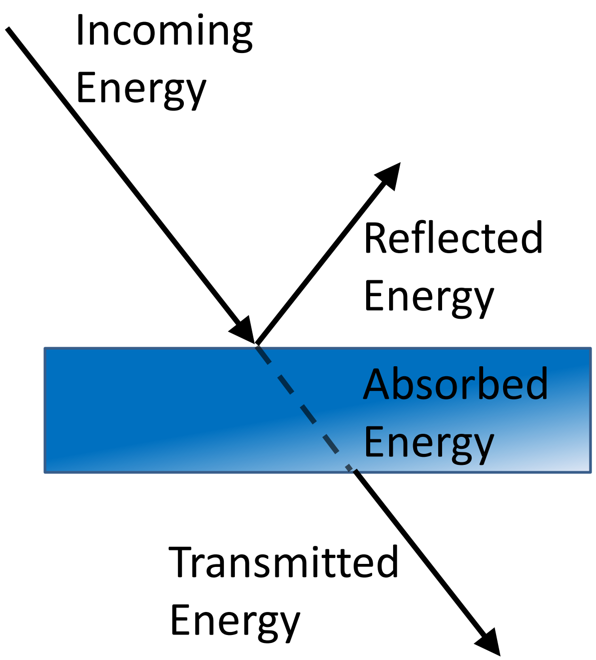
Planck Blackbody Curve for 300 K

Diffuse and Specular Reflection

We saw previously that when energy strikes a surface, it may be absorbed, transmitted, or reflected;

For our discussion, we're going to consider only opaque surfaces;

In this case, transmitted radiation is zero.



Diffuse and Specular Reflection

Reflection may occur in a number of ways:

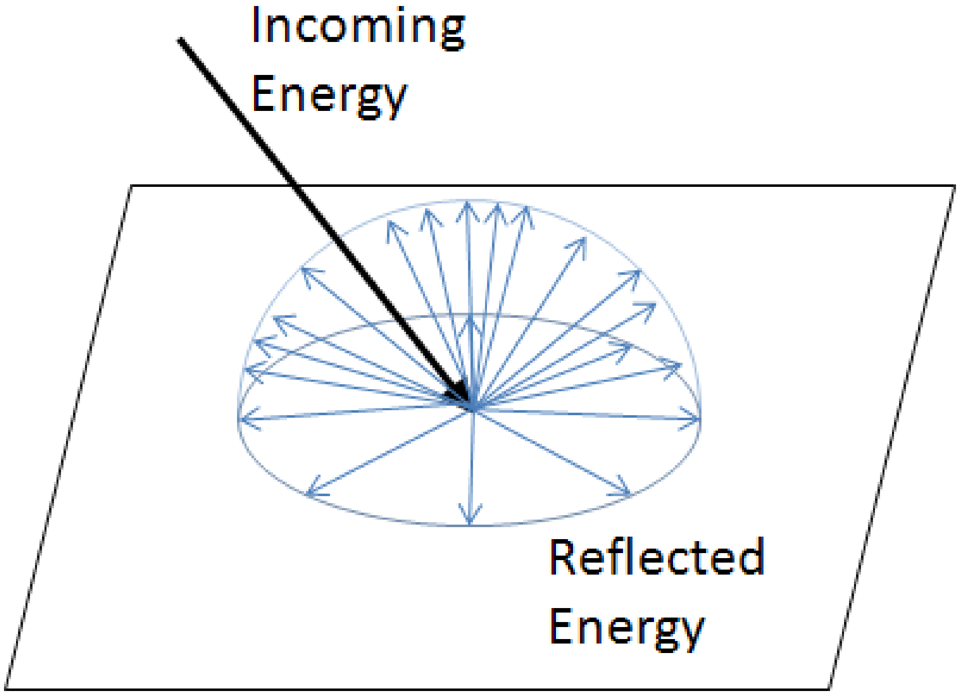
Diffuse -- energy striking a surface is scattered in all directions;

Specular -- the energy angle of incidence is equal to the angle of reflection in the pure case;

Mixed -- a combination of both, diffuse and specular modes.

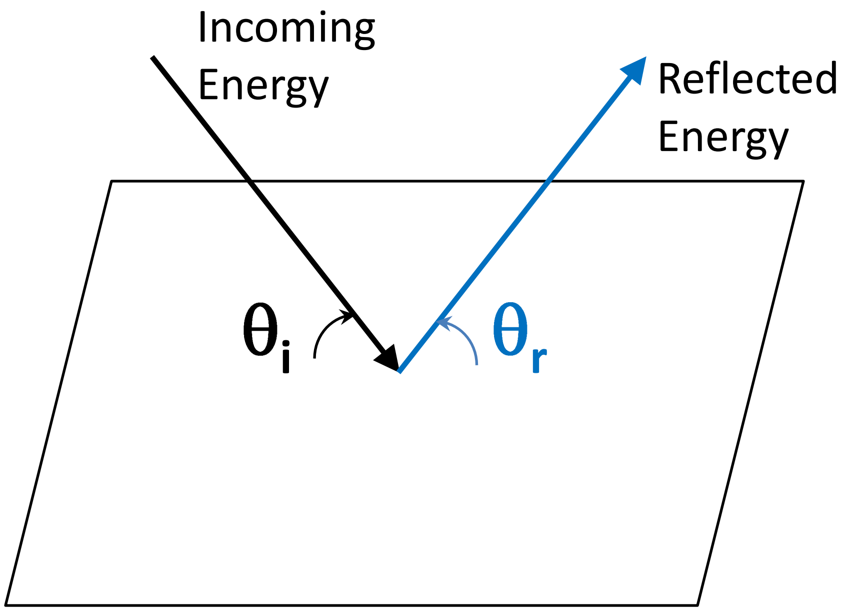
Diffuse and Specular Reflection

Diffuse -- energy striking a surface is scattered in all directions.



Diffuse and Specular Reflection

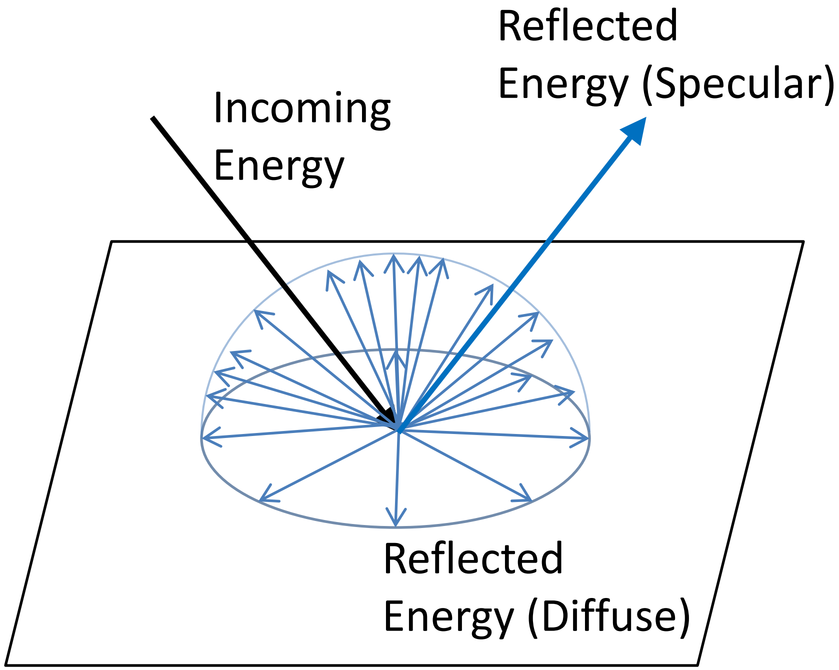
Specular -- In a pure specular reflection, the angle of incidence (θ_i) is equal to the angle of reflection (θ_r).



Diffuse and Specular Reflection

Mixed-- A combination of, both, diffuse and specular reflections;

In this case, we consider a **specular fraction** -- the fraction of the total reflected energy that is reflected specularly.

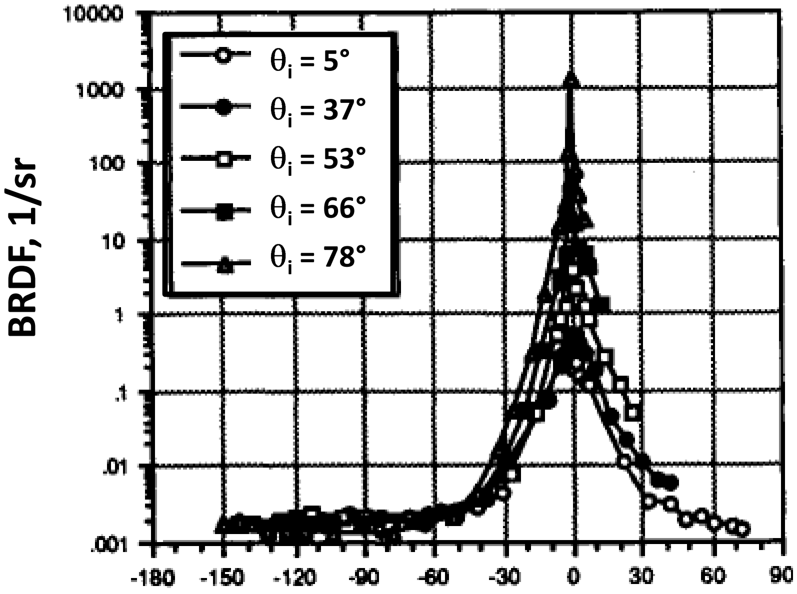


Real Surfaces

Reflection from real surfaces is often more complex than the pure diffuse or specular case;

Bidirectional reflectance distribution function (BRDF) (Adapted from Ref. 2).

S13-GLO Painted Aluminum



Angle from Specular In Incidence Plane, deg

Wavelength = 10.63 μm

What is a Radk and What is its Role in Heat Transfer?

Conduction, Convection and Radiation

Heat is transferred by three distinct modes:

Conduction;

Convection;

Radiation.

The Thermal Network

The goal of thermal analysis is to represent, accurately, the physics of heat transfer phenomena;

Nodes represent objects with mass (diffusion), objects that can reasonably be modeled as massless (arithmetic), and substantial heat sinks (boundary);

Conductors represent heat flow paths.

The Thermal Network

Conductors come in the following varieties:

Conduction Conductor -- a heat transfer path between two solid objects;

Convection Conductor -- a heat transfer path between a solid object and a convecting liquid or gas;

Radiation Conductor -- a heat transfer path, via electromagnetic radiation, between two objects.

The Thermal Network

Heat transfer via conduction and convection is proportional to *ΔT*:

$$\dot{Q}_{cond} = G_{cond}\Delta T = \frac{kA}{L}\Delta T$$

$$\dot{Q}_{conv} = G_{conv}\Delta T = hA\Delta T$$

But, heat transfer via radiation takes this form:

$$\dot{Q}_{rad} = G_{rad}\Delta(T^4)$$

The Thermal Network

Differential equation solvers for thermal analysis solve *linear* differential equations;

The introduction of radiation into thermal networks adds non-linearity to the problem;

But, the radiation portion of the thermal network can be linearized to make solution as part of the overall network possible.

The Thermal Network (Ref. 3)

We seek to express the radiation heat transfer between the two nodes of interest in terms of ΔT.

$$\begin{aligned}\left(\dot{Q}_{rad}\right)_{12} &= G_{rad}\left(T_1^4 - T_2^4\right)\\ &= G_{rad}\left(T_1^2 + T_2^2\right)\left(T_1^2 - T_2^2\right)\\ &= G_{rad}\left(T_1^2 + T_2^2\right)\left(T_1 + T_2\right)\left(T_1 - T_2\right)\\ &= \boxed{G_{rad}\left(T_1^2 + T_2^2\right)\left(T_1 + T_2\right)}\Delta T\end{aligned}$$

Linearized Radiation Conductor

The Thermal Network

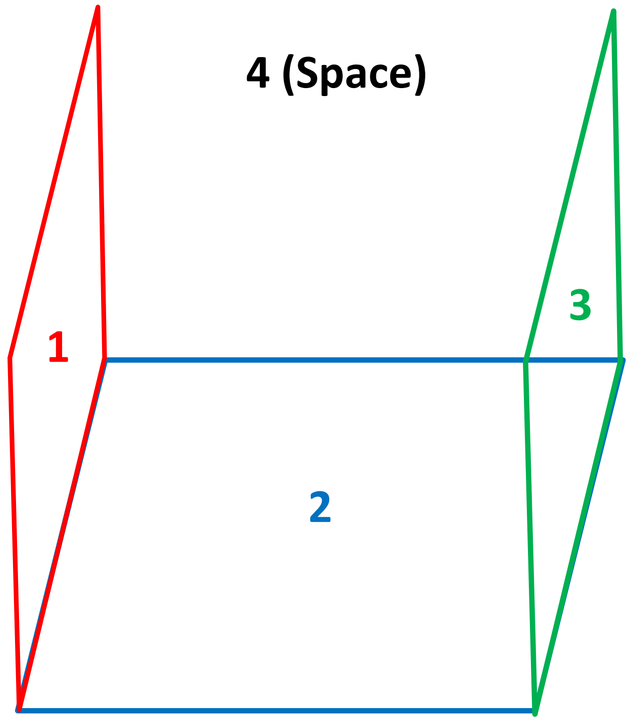
But what about the G_{rad} term?

This is the *radiation conductor*, or, **Radk** for short;

Developing the Radk, via form factors and grey body factors, will be the focus of the remainder of this lesson.

The Thermal Network

Consider the following geometry:



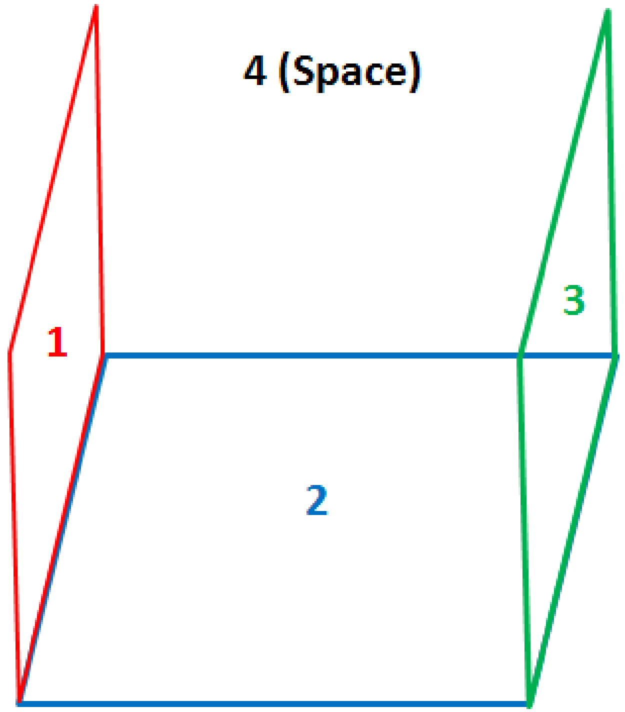
The Thermal Network

Considering *only* conduction between the objects:

1 and 2 can exchange energy;
2 and 3 can exchange energy.

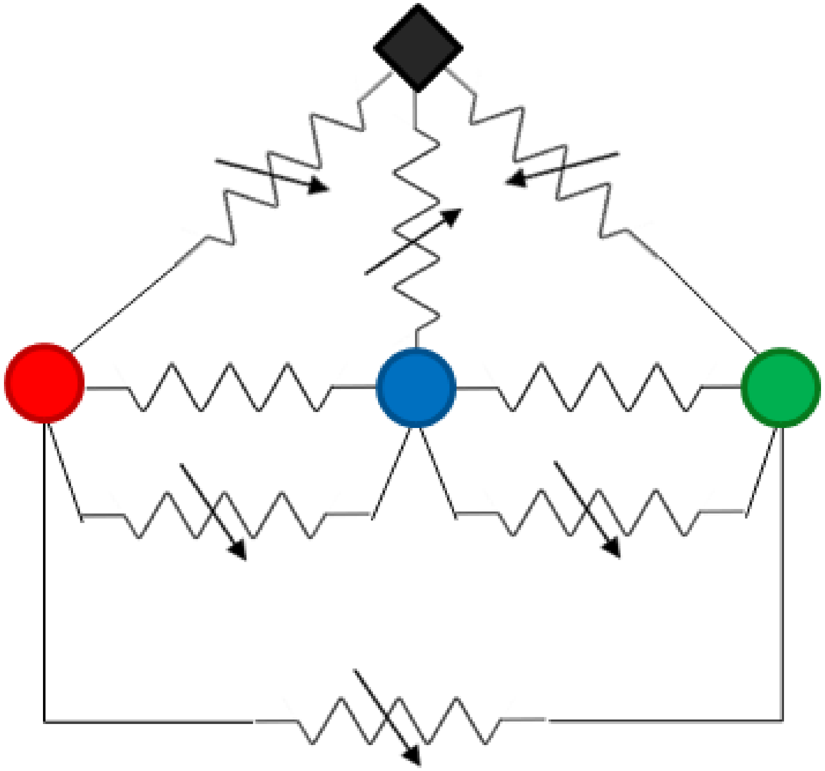
When we *add* radiation:

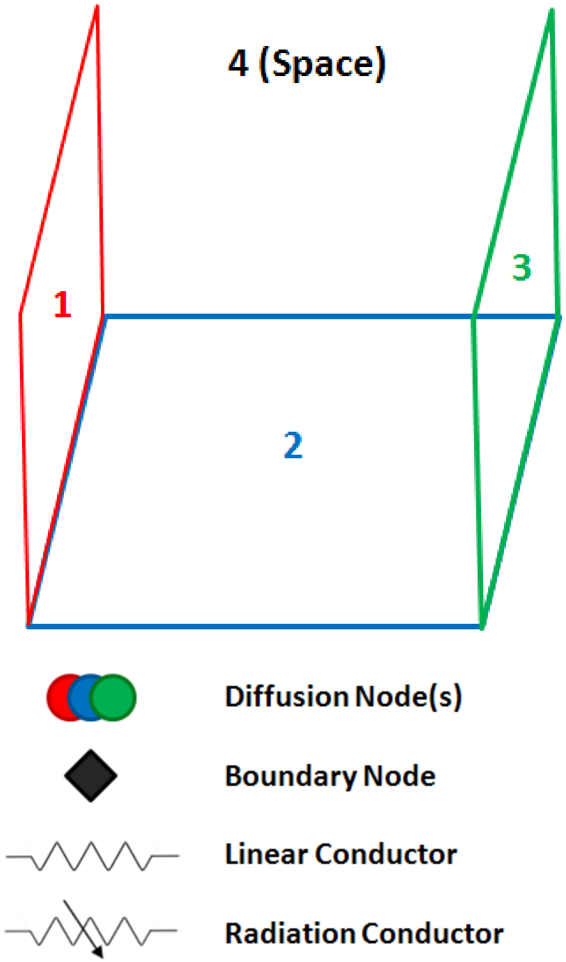
1 and 2 can exchange energy;
1 and 3 can exchange energy;
1 and **4** can exchange energy;
2 and 3 can exchange energy;
2 and **4** can exchange energy;
3 and **4** can exchange energy.



The Thermal Network

The thermal network looks like this:

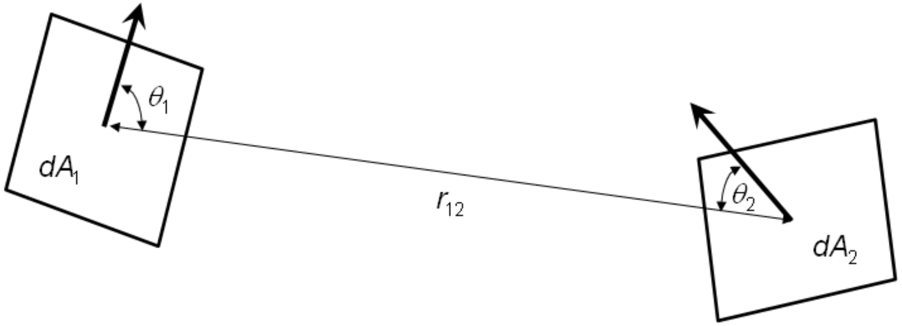




Form Factors

The Form Factor

A **form factor** (a.k.a. shape factor, configuration factor, view factor) describes how well one object can "see" another object;



$$FF_{12} = \frac{1}{A_1} \iint \frac{\cos\theta_1 \cos\theta_2}{\pi r_{12}^2} dA_1 dA_2$$

The form factor may take on a value from zero to unity.

The Form Factor

A form "factometer" was used in conjunction with a physical model to determine key form factors for radiation analysis; the technique used a partial spherical mirror and the projection of the reflected image of a target surface on the unit circle;

With modern computers, however, calculating form factors is now an analytical pursuit.

Exact Solutions

Throughout this lesson, we'll be exploring techniques for approximating the form factor between two surfaces;

There are a limited number of geometries for which exact solutions exist;

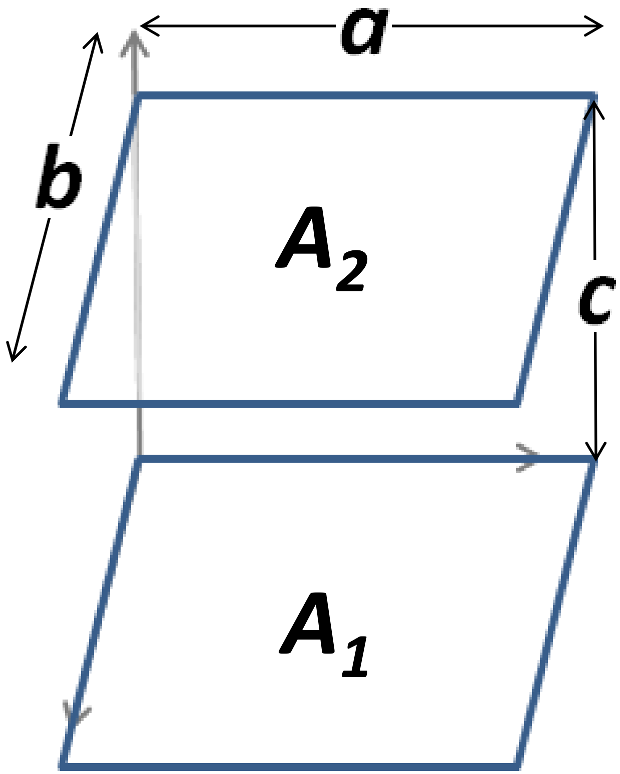
Having an exact solution available for comparison will help us show the benefit of the approximate methods.

Exact Solutions

We'll be using the geometry of parallel unit plates with unit separation throughout this lesson;

Let's take a look at the exact solution for this geometry.

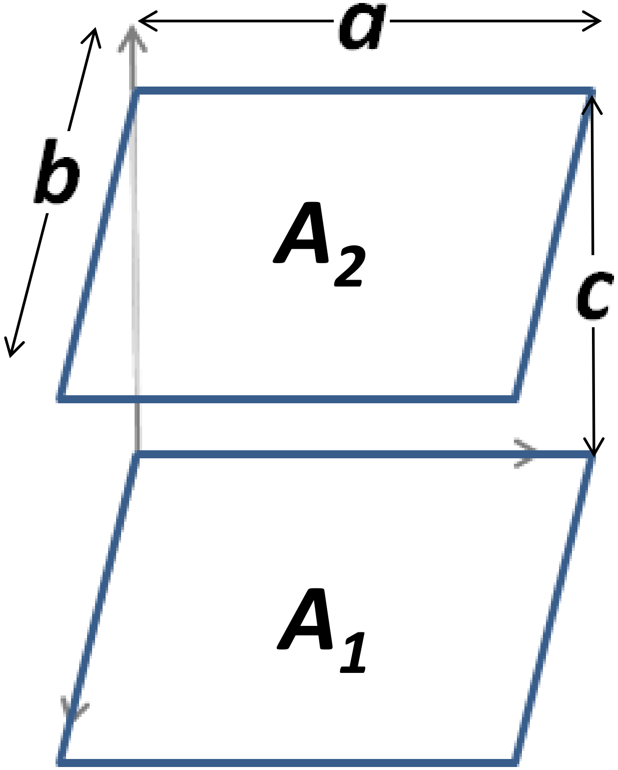
For convenience, we define: *X = a/c* and *Y = b/c*



Exact Solutions (Ref. 4)

An *exact* solution is possible through a technique called contour integration:

$$FF_{12} = \frac{2}{\pi xy}\left\{ln\left[\frac{(1+X^2)(1+Y^2)}{1+X^2+Y^2}\right]^{1/2} + X\sqrt{1+Y^2}tan^{-1}\left(\frac{X}{\sqrt{1+Y^2}}\right) + Y\sqrt{1+X^2}tan^{-1}\left(\frac{Y}{\sqrt{1+X^2}}\right) - Xtan^{-1}X - Ytan^{-1}Y\right\}$$



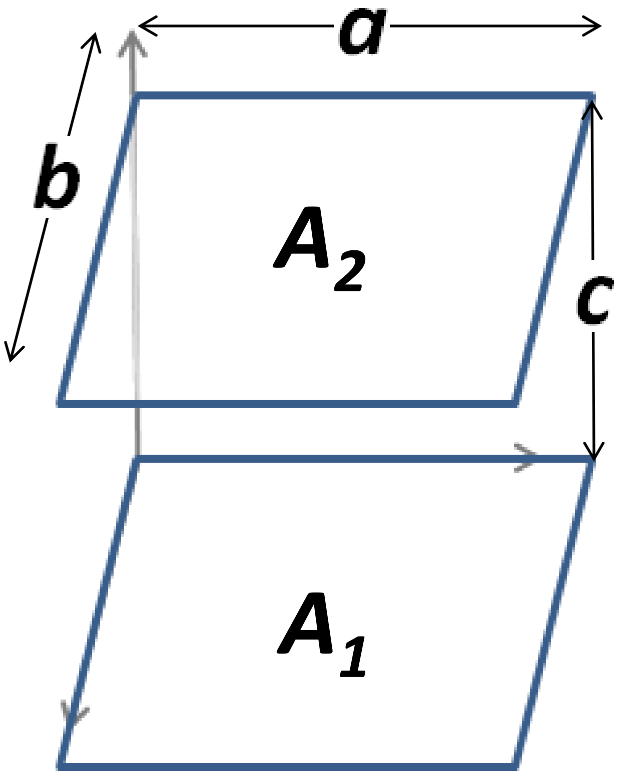
Exact Solutions

For unit plates with unit separation:

$X = Y = 1$

and the exact solution for FF_{12} becomes:

$FF_{12} = 0.2$



The Form Factor

Form factors may be calculated using a variety of techniques including:

Double area summation;
Nusselt Sphere technique;
Crossed-String method;
Monte Carlo ray tracing;
Contour integration;
Hemi-cube.

We'll examine the **highlighted** techniques in detail.

The Double Area Summation Method

Double Area Summation

Consider, again, the integral defining the form factor:

$$FF_{12} = \frac{1}{A_1} \iint \frac{\cos\theta_1 \cos\theta_2}{\pi r_{12}^2} dA_1 dA_2$$

We see that finite area approximations may be made and summations may be substituted for the integrals:

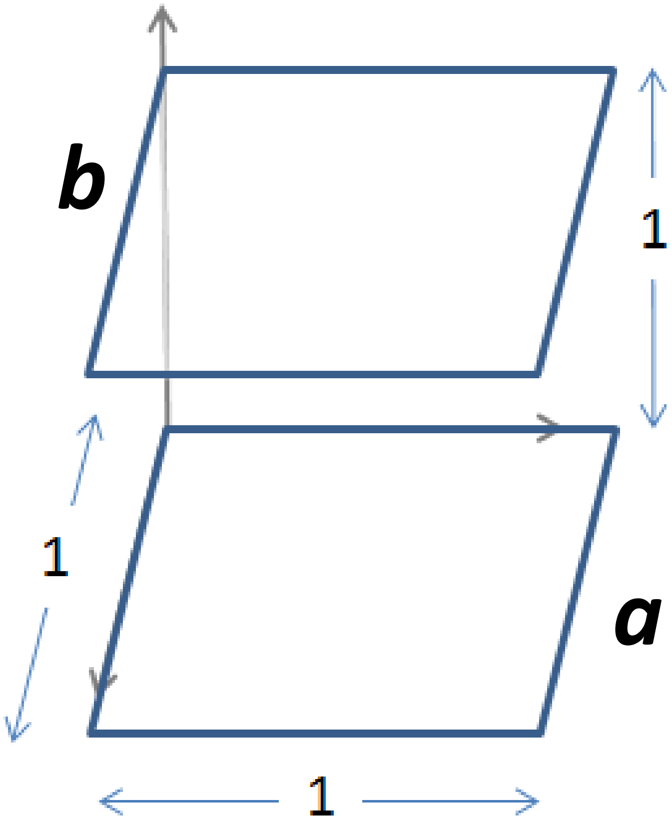
$$FF_{ab} = \frac{1}{A_a} \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{cos\theta_i cos\theta_j}{\pi r_{ij}^2} \Delta A_i \Delta A_j$$

Double Area Summation Example

Consider two parallel unit plates separated by one unit;

We seek the form factor between surfaces ***a*** and ***b***.

We'll examine a number of solutions.

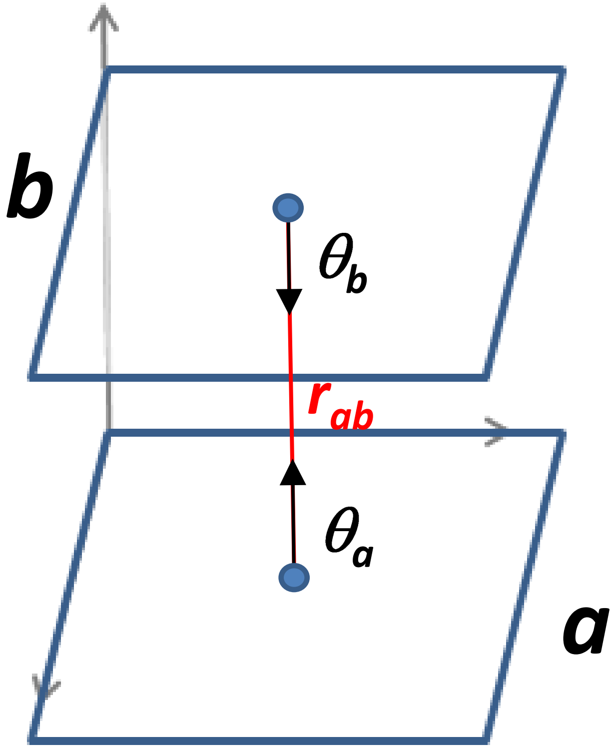


Double Area Summation Example

First, consider the case where each surface consists of one element ($n = 1, m = 1$);

The angles (θ_a and θ_b) between the line connecting the element centers and the surface normal is 0°;

The distance (r_{ab}) between element center is 1.

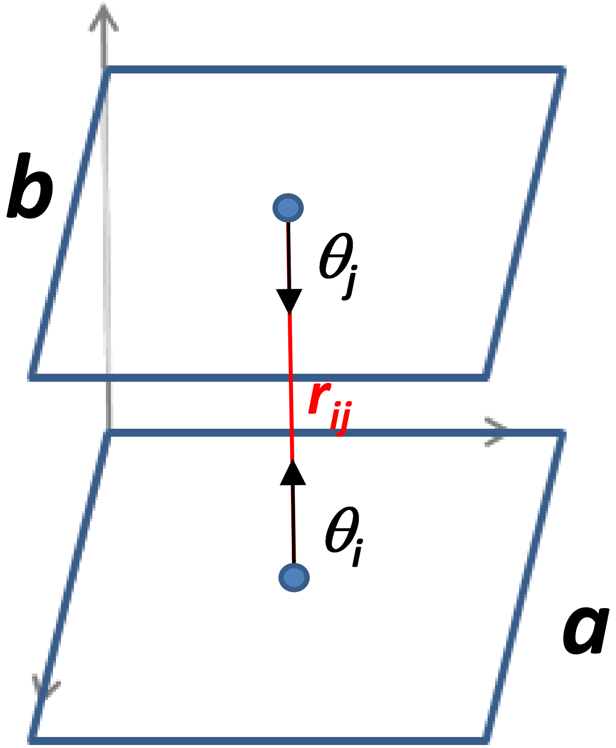


Double Area Summation Example

Applying the formula:

$$FF_{ab} = \frac{1}{A_a}\sum_{i=1}^{n}\sum_{j=1}^{m}\frac{cos\theta_i cos\theta_j}{\pi r_{ij}^2}\Delta A_i \Delta A_j$$

$$FF_{ab} = \frac{1}{(1)}\sum_{i=1}^{1}\sum_{j=1}^{1}\frac{cos(0°)cos(0°)}{\pi(1)^2}(1)(1)$$



Our estimate is:

$$FF_{ab} = 0.31831$$

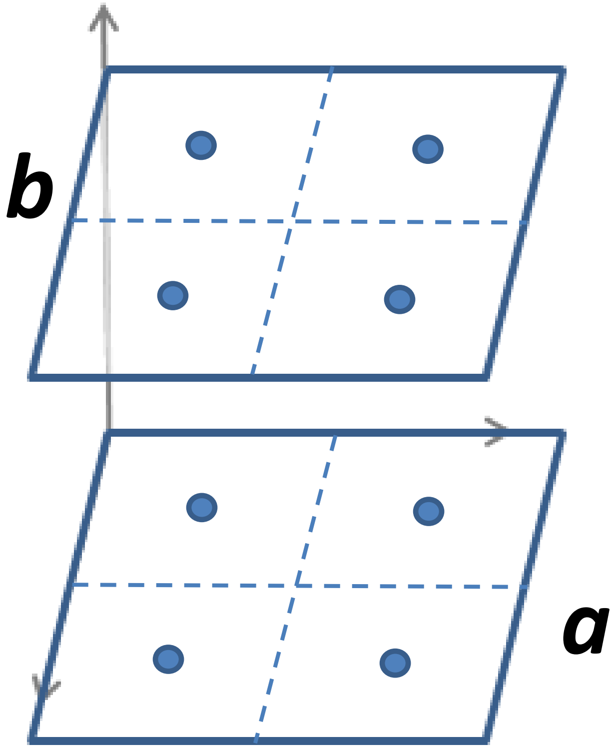
Recall, the exact solution is 0.2 -- we can do better.

Double Area Summation Example

Let's subdivide each element into 4 subelements (n = 4, m = 4);

The same calculation is performed for each subelement on the other surface;

A total of 16 calculations is required.

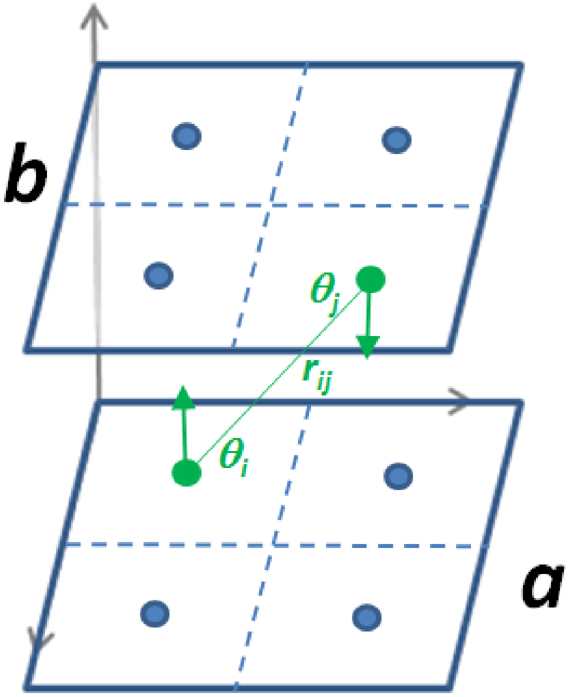


Double Area Summation Example

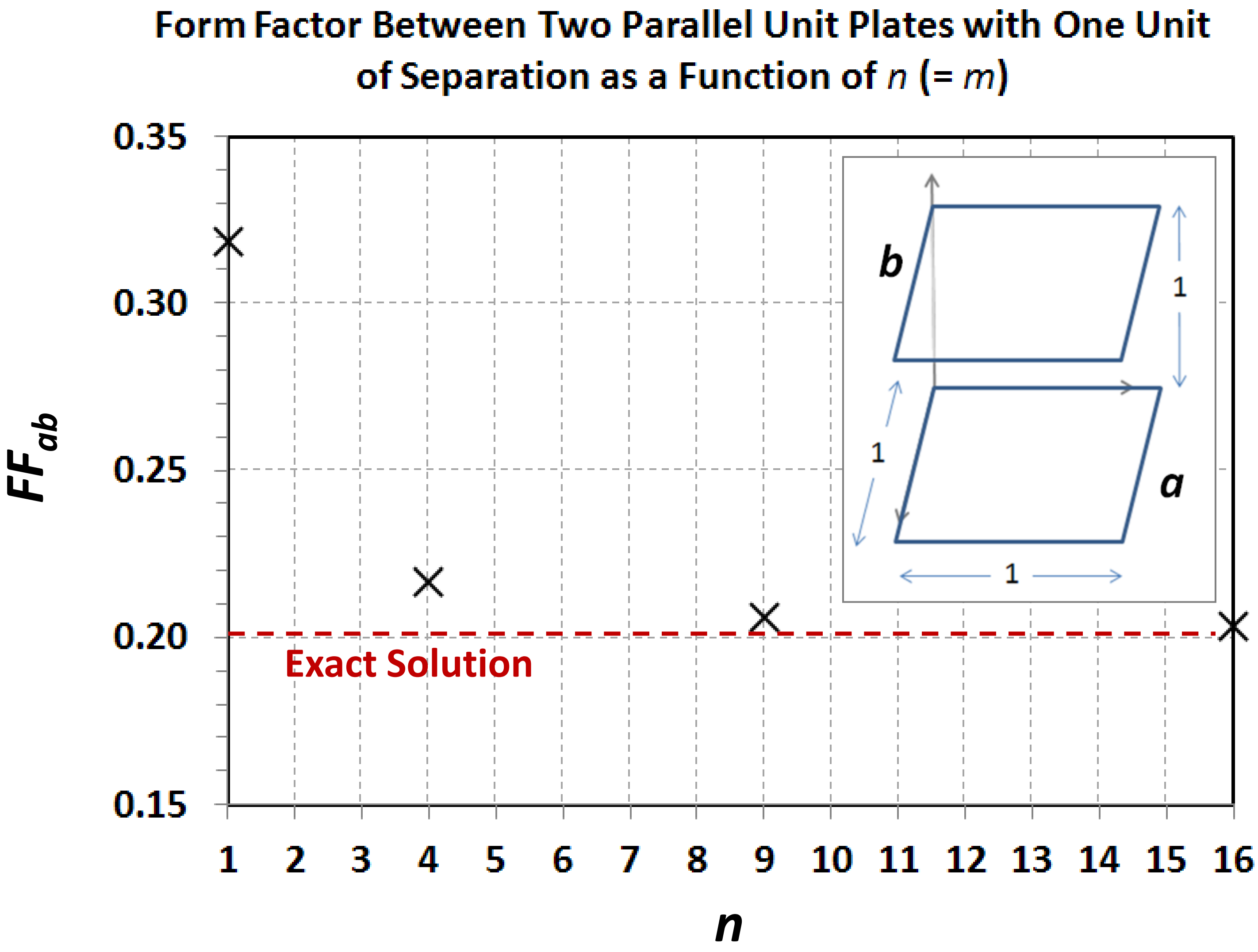
The subelement to subelement calculations are :

Plate *a*	Plate *b*	r_{ij}	$\cos(\theta_i)$	$\cos(\theta_j)$	FF_{ij}
1	1	1.000	1.000	1.000	0.020
1	2	1.118	0.894	0.894	0.013
1	3	1.118	0.894	0.894	0.013
1	4	1.225	0.816	0.816	0.009
2	1	1.118	0.894	0.894	0.013
2	2	1.000	1.000	1.000	0.020
2	3	1.225	0.816	0.816	0.009
2	4	1.118	0.894	0.894	0.013
3	1	1.118	0.894	0.894	0.013
3	2	1.225	0.816	0.816	0.009
3	3	1.000	1.000	1.000	0.020
3	4	1.118	0.894	0.894	0.013
4	1	1.225	0.816	0.816	0.009
4	2	1.118	0.894	0.894	0.013
4	3	1.118	0.894	0.894	0.013
4	4	1.000	1.000	1.000	0.020

FF_{ab} 0.217



Double Area Summation Example

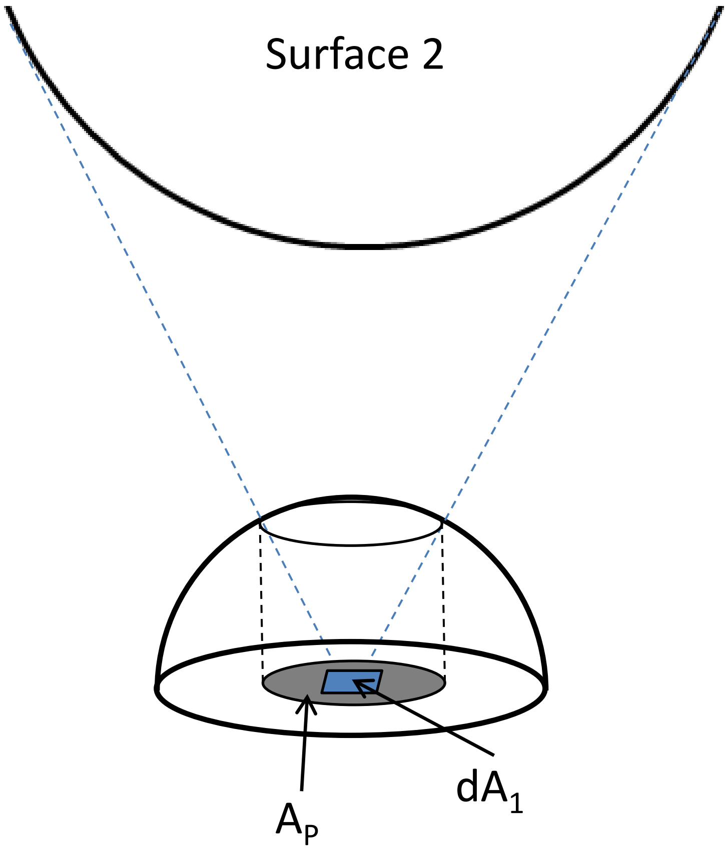


The Nusselt Sphere Method

Nusselt Sphere

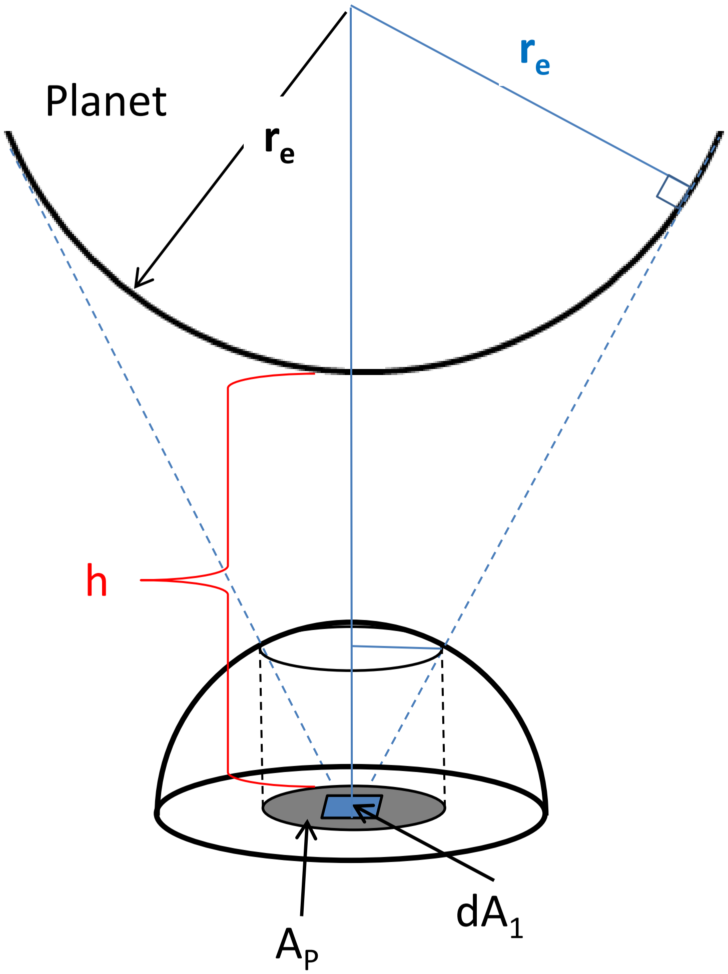
The Nusselt Sphere technique is one of many ways to calculate form factors;

The form factor from dA_1 to Surface 2 is calculated as the projected area, A_P, divided by the area of the hemisphere's circular base.



Nusselt Sphere

Let's use the Nusselt Sphere technique for calculating the form factor to the planet from an orbiting plate, at altitude h above the planet, whose surface normal faces the nadir direction.

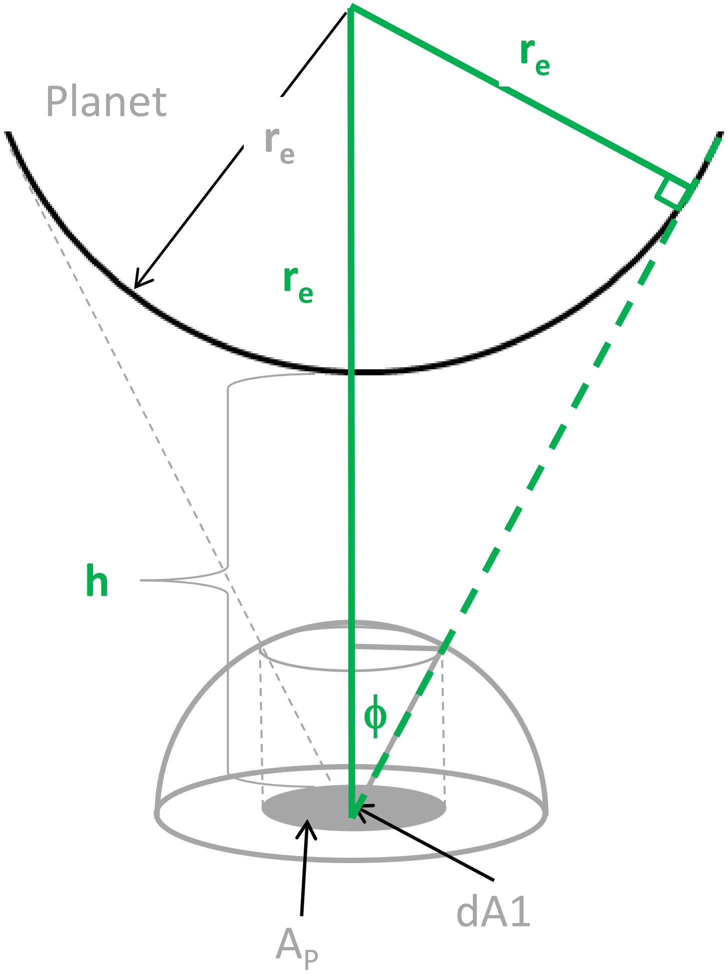


Nusselt Sphere

We see that we can construct a right triangle (in green) with a short side measuring r_e and a hypotenuse measuring r_e+h;

We define the angle ϕ by noting:

$$\sin \phi = \frac{r_e}{r_e + h}$$

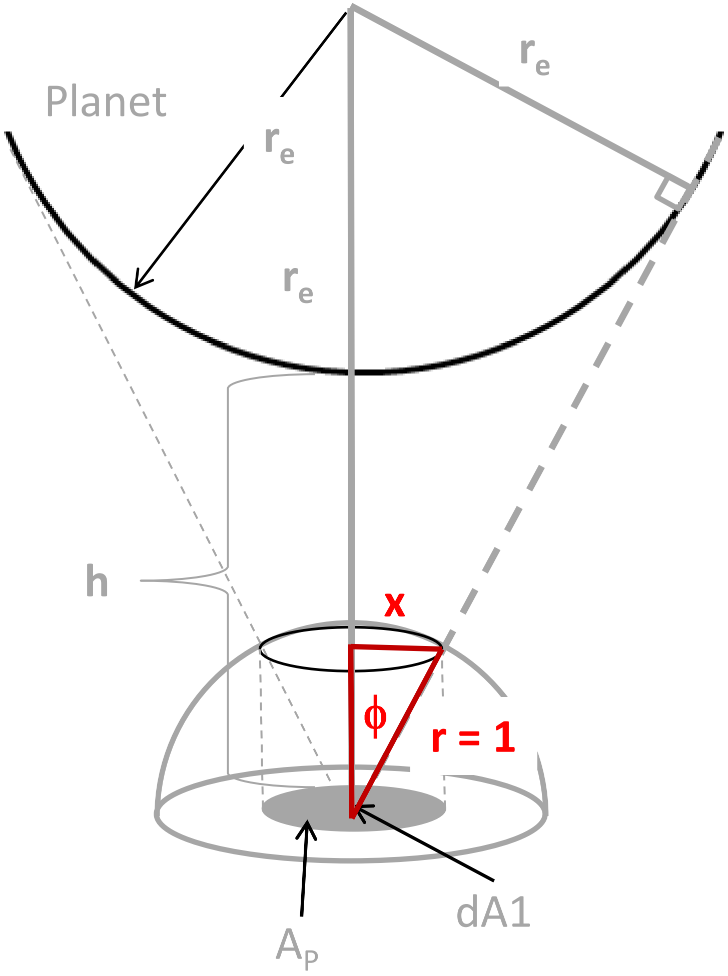


Nusselt Sphere

Similarly, we can construct a right triangle (in red) with a hypotenuse measuring unity and the angle ϕ, already defined;

We define the distance *x* by noting:

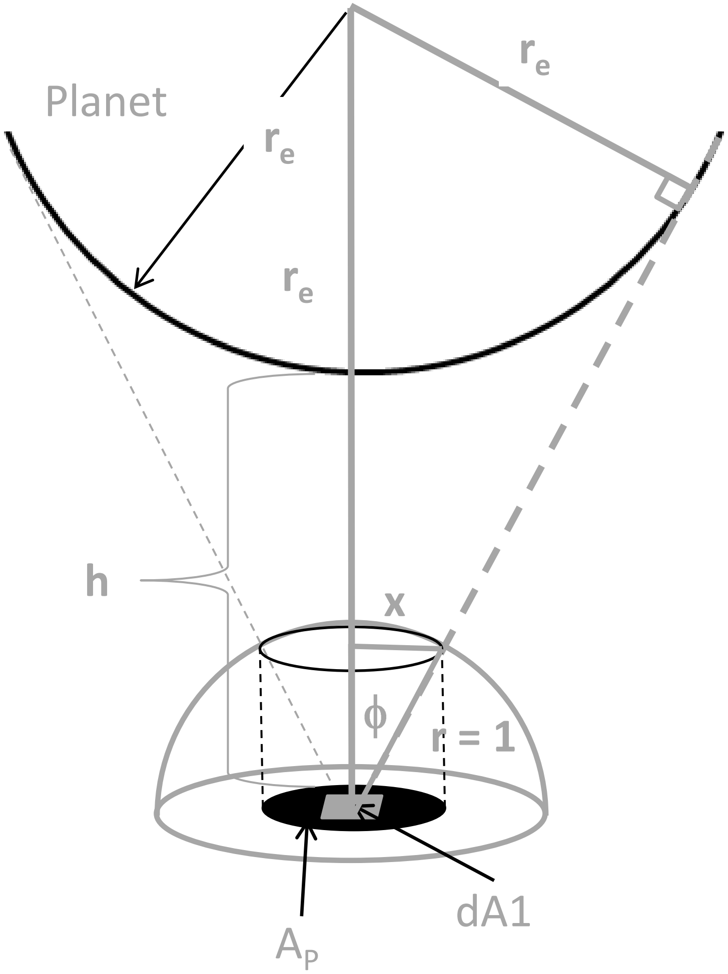
$$\sin\phi = \frac{x}{r} = \frac{x}{1} = x = \frac{r_e}{r_e + h}$$



Nusselt Sphere

Projecting *x* down to the base, we see that the ratio of the projected circular area to the total area of the base is:

$$FF = \frac{\pi x^2}{\pi 1^2} = \left(\frac{r_e}{r_e + h}\right)^2$$



The Crossed-String Method

Crossed-String Method (Ref. 5)

The Crossed-String method may be used on two dimensional geometries of infinite extent;

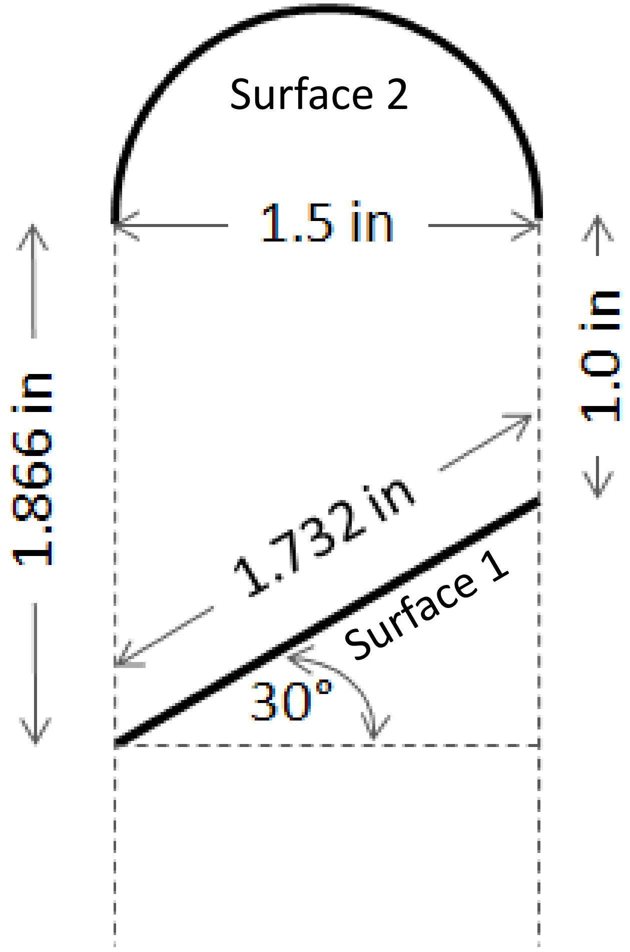
The general formula for calculating a form factor between two surfaces is:

$$FF_{ij} = \frac{\sum lengths\ of\ crossed\ strings - \sum lengths\ of\ uncrossed\ strings}{2 \times length\ of\ string\ on\ i}$$

Crossed-String Method

Consider the following two-dimensional geometry of infinite extent;

We seek the form factor between surface 1 and surface 2.



Crossed-String Method

Let's define our string lengths:

$L_1 = 1.732$ in

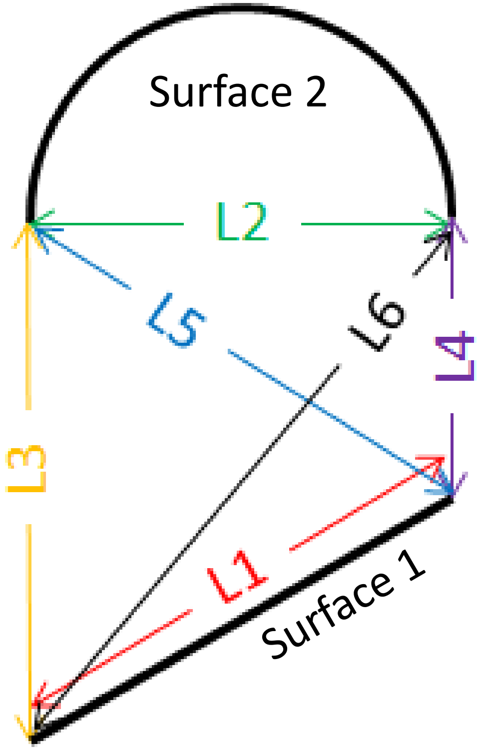
$L_2 = 1.5$ in

$L_3 = 1.866$ in

$L_4 = 1.0$ in

$L_5 = 1.803$ in

$L_6 = 2.394$ in



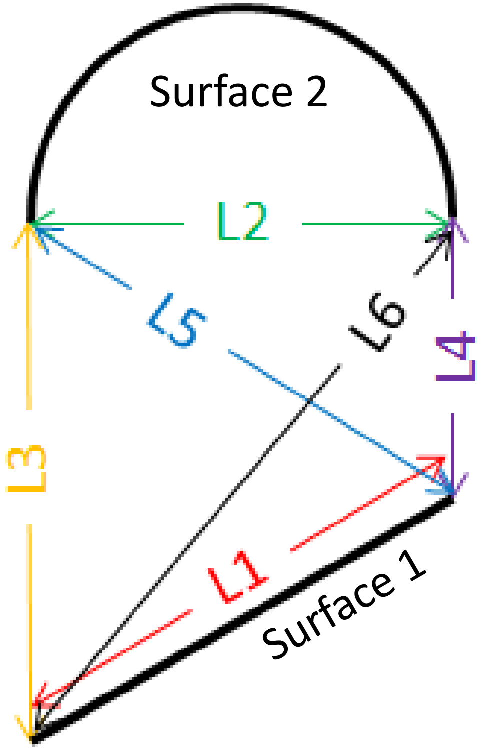
Crossed-String Method

Solving...

$$FF_{12} = \frac{(L_5 + L_6) - (L_3 + L_4)}{2L_1}$$

$$FF_{12} = \frac{(1.803 + 2.394) - (1.866 + 1.0)}{2(1.732)}$$

$$FF_{12} = 0.384$$



The Monte Carlo Ray Tracing Method

Monte Carlo Methods

A Monte Carlo method uses *random numbers* to perform an integration;

How can a random number can be used to accomplish such a feat?

Furthermore, how can we get usable answers from anything involving random numbers?

Let's start with an example.

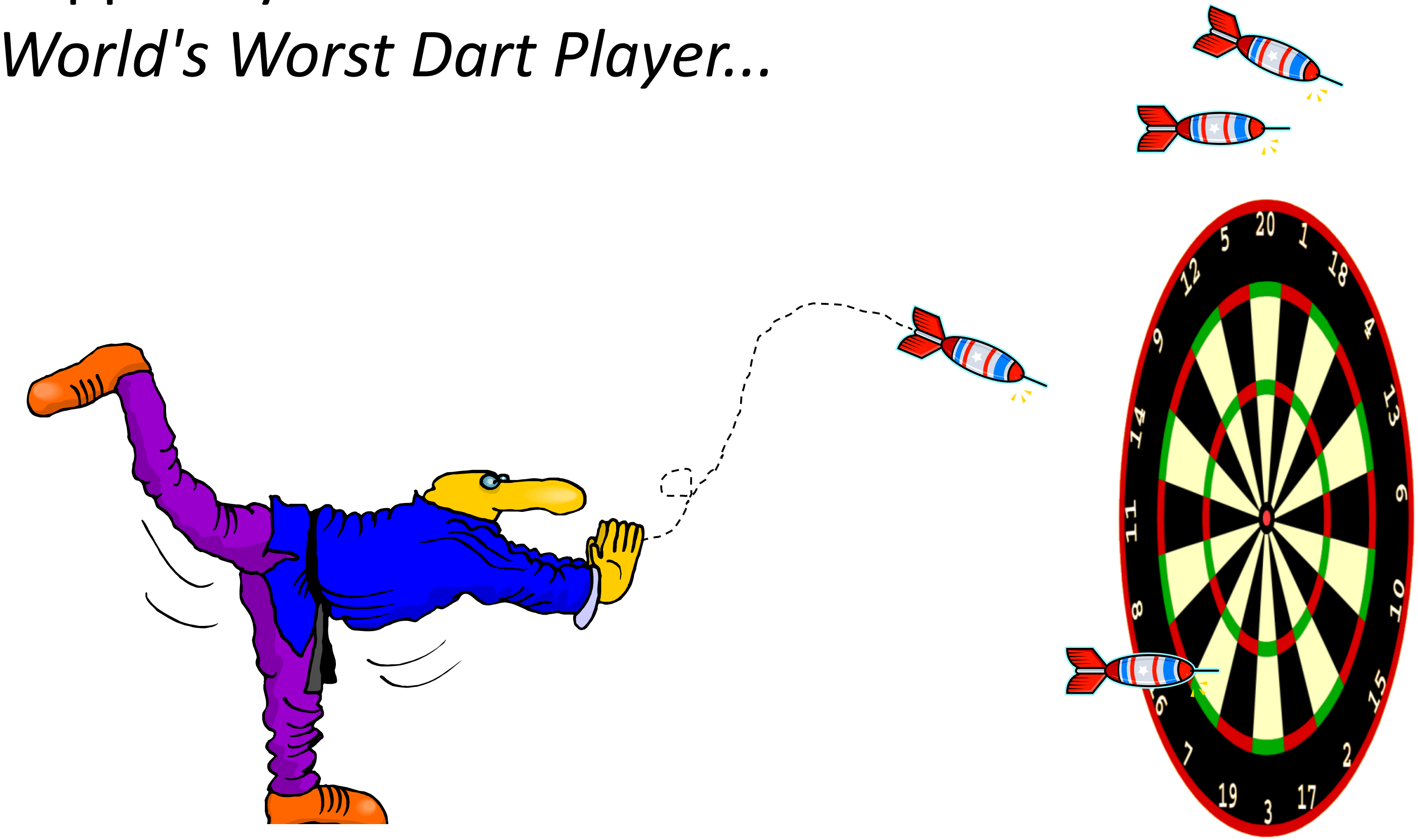
Example -- The "World's Worst Dart Player"

To see how random numbers can be applied in integration, let's consider the challenge of approximating π;

We'll establish a thought experiment, establish some equations to help us conduct this experiment, and then use random numbers to help us achieve our goal.

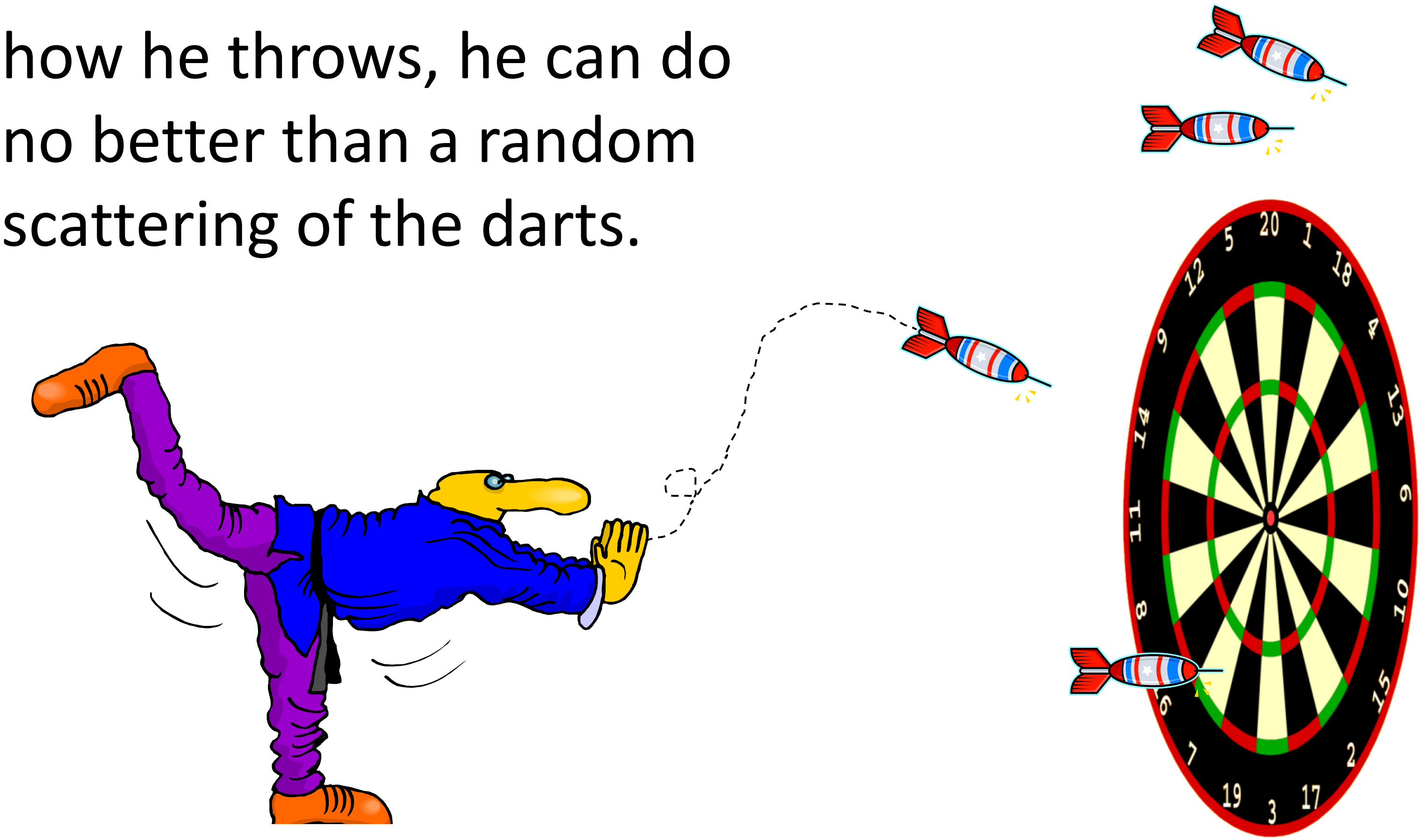
Example -- The "World's Worst Dart Player"

Suppose you encounter the
World's Worst Dart Player...



Example -- The "World's Worst Dart Player"

He's so bad that no matter how he throws, he can do no better than a random scattering of the darts.



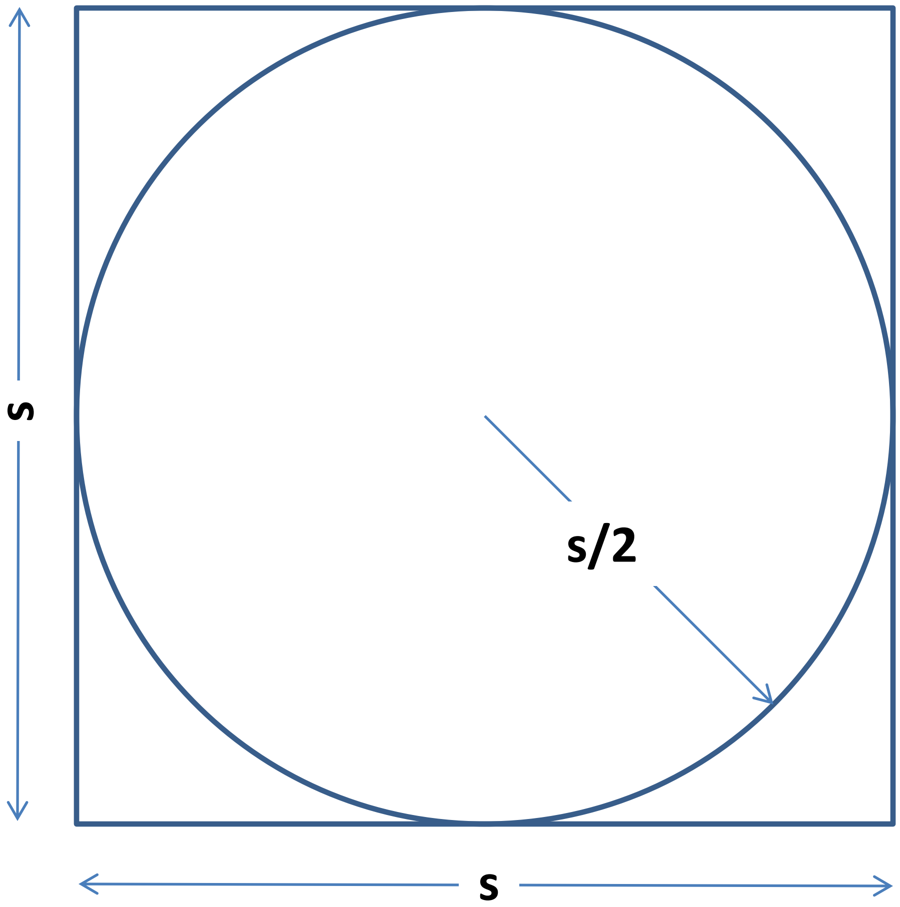
Example -- The "World's Worst Dart Player"

So, to make the game fair, you decide to build a very large, circular dart board -- one that just fits onto a square wall;

Our dart player may be the world's worst but he can certainly hit somewhere on the square wall!

Example -- The "World's Worst Dart Player"

Let's put some dimensions on our wall and dartboard -- represented here by a square and circle, respectively.



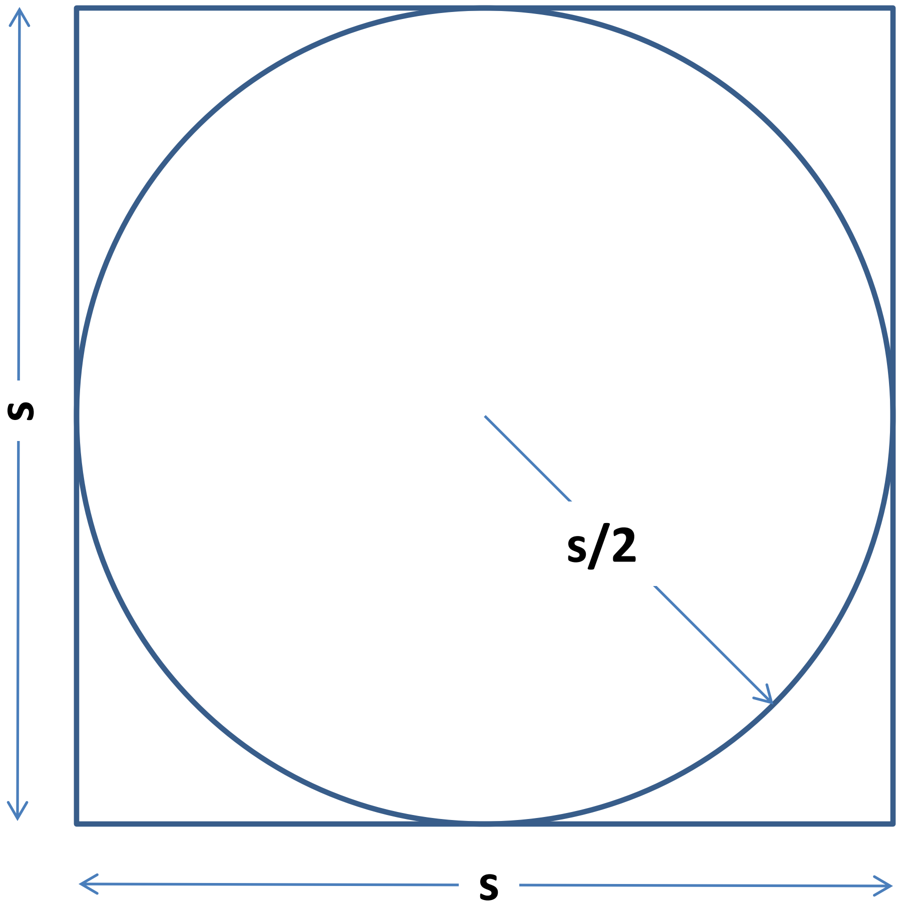
Example -- The "World's Worst Dart Player"

The area of the square is given by:

$$A_s = s^2$$

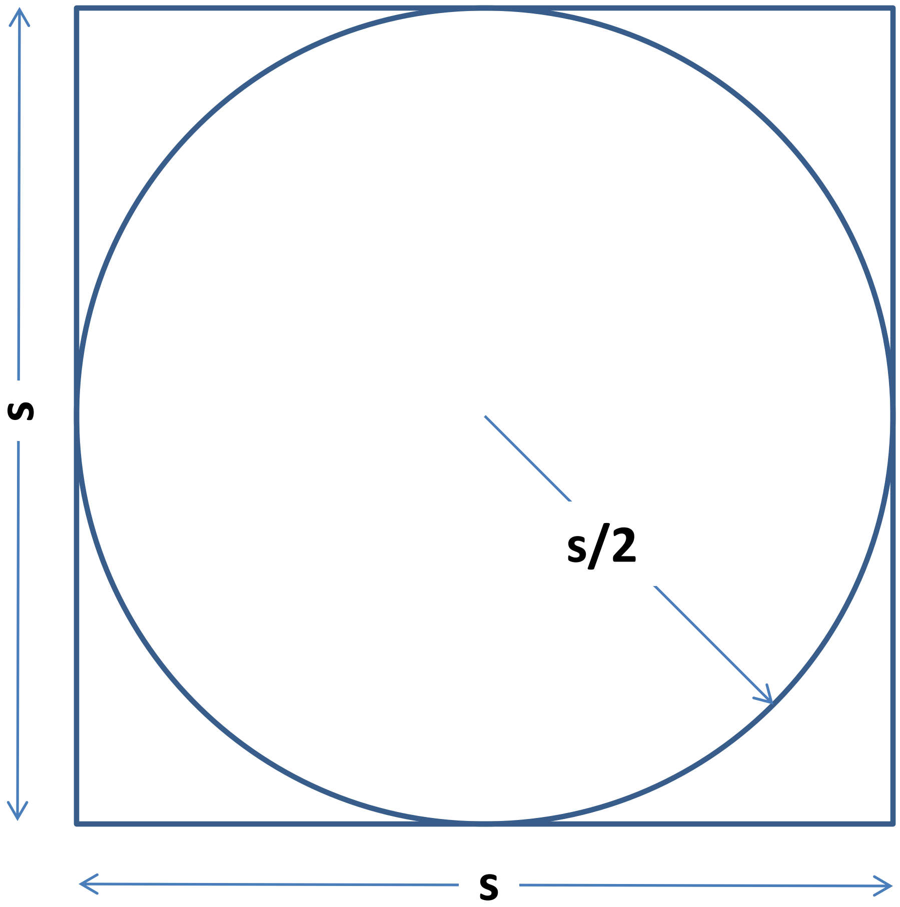
And the area of the circle is, simply:

$$A_c = \pi s^2/4$$

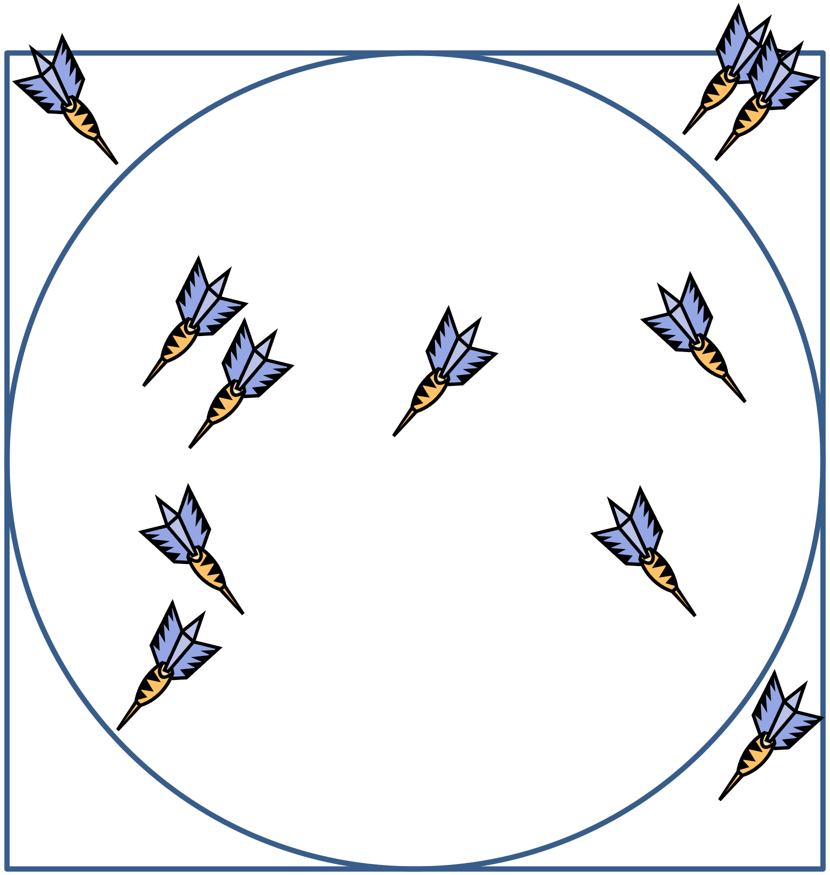


Example -- The "World's Worst Dart Player"

If all of our dart thrower's darts must land *somewhere* inside the square, the chances that a dart will land inside the circle will depend on the relative areas of the circle and the square.



Example -- The "World's Worst Dart Player"



In other words, for a large *total* number of darts, n_t , we have:

$$^{n_c}/_{n_t} \approx {}^{A_c}/_{A_s}$$

where n_c is the number of darts that land *inside* the circle.

Example -- The "World's Worst Dart Player"

But, we have expressions for areas A_c and A_s:

$$ {}^{n_c}/_{n_t} \approx {}^{\left({}^{\pi \cancel{s^2}}/_{4}\right)}\Big/_{\cancel{s^2}} = {}^{\pi}/_{4} $$

or...

$$ \pi \approx 4\,{}^{n_c}/_{n_t} $$

This gives us a framework for our numerical experiment.

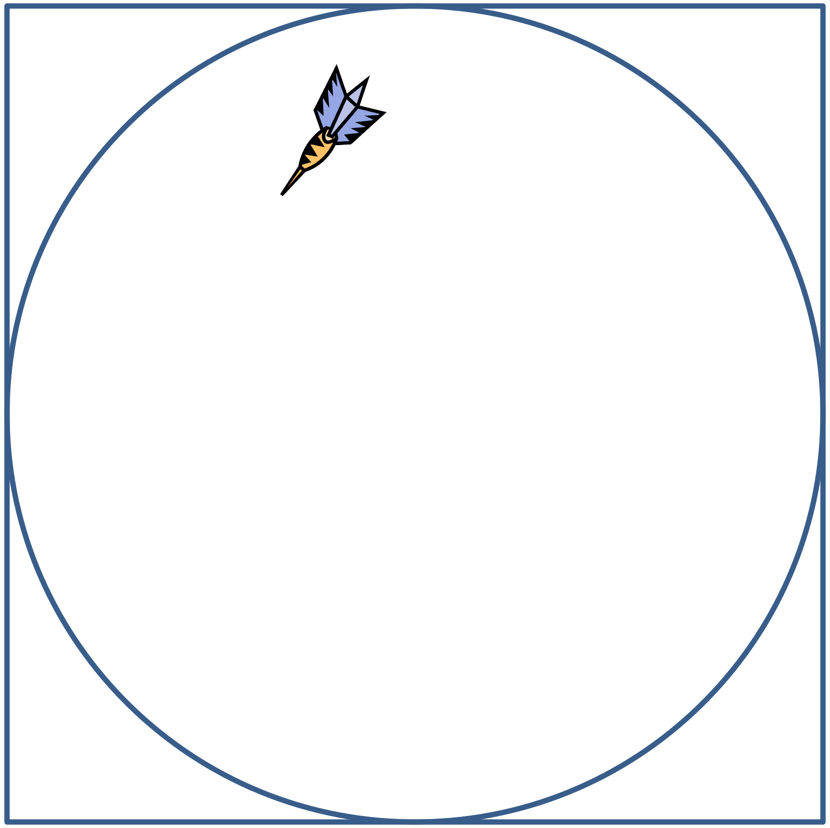
Example -- The "World's Worst Dart Player"

Let's throw our first dart;

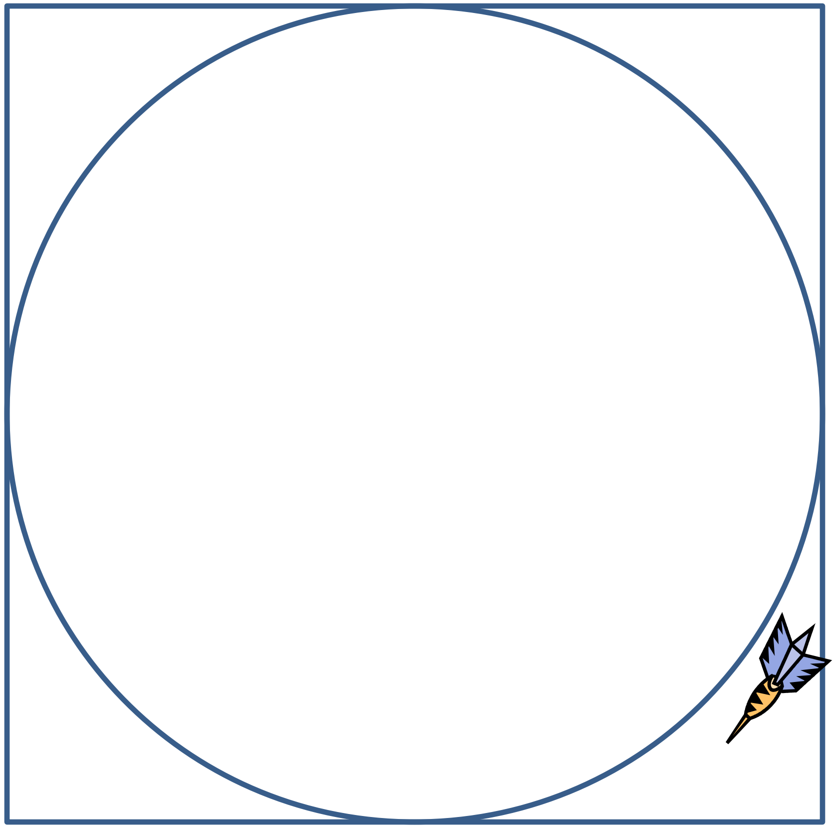
If it lands *inside* the circle, our estimate of π for $n_t = 1$ is:

$$\pi \cong 4\ (1/1) = 4$$

Not bad for one dart.



Example -- The "World's Worst Dart Player"



But since it's a random dart, it could have just as easily landed *outside* the dartboard;

Our estimate becomes:

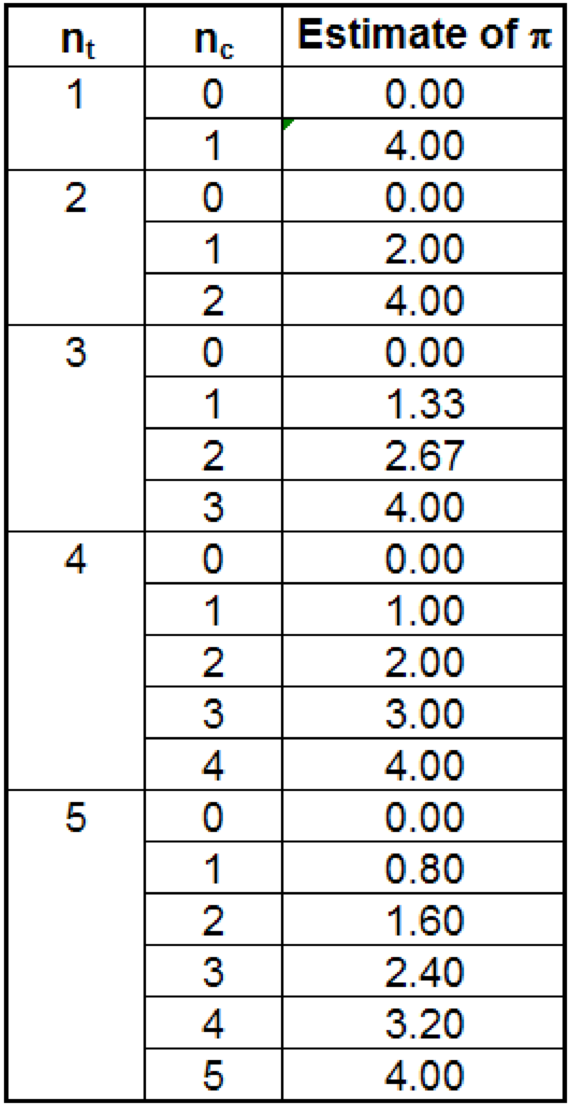
$$\pi \cong 4\ (0/1) = 0$$

Not so good.

Example -- The "World's Worst Dart Player"

Depending on where the darts fall for each n_t, a number of different outcomes are possible;

As n_t increases, more and more n_c/n_t combinations are possible.



n_t	n_c	Estimate of π
1	0	0.00
	1	4.00
2	0	0.00
	1	2.00
	2	4.00
3	0	0.00
	1	1.33
	2	2.67
	3	4.00
4	0	0.00
	1	1.00
	2	2.00
	3	3.00
	4	4.00
5	0	0.00
	1	0.80
	2	1.60
	3	2.40
	4	3.20
	5	4.00

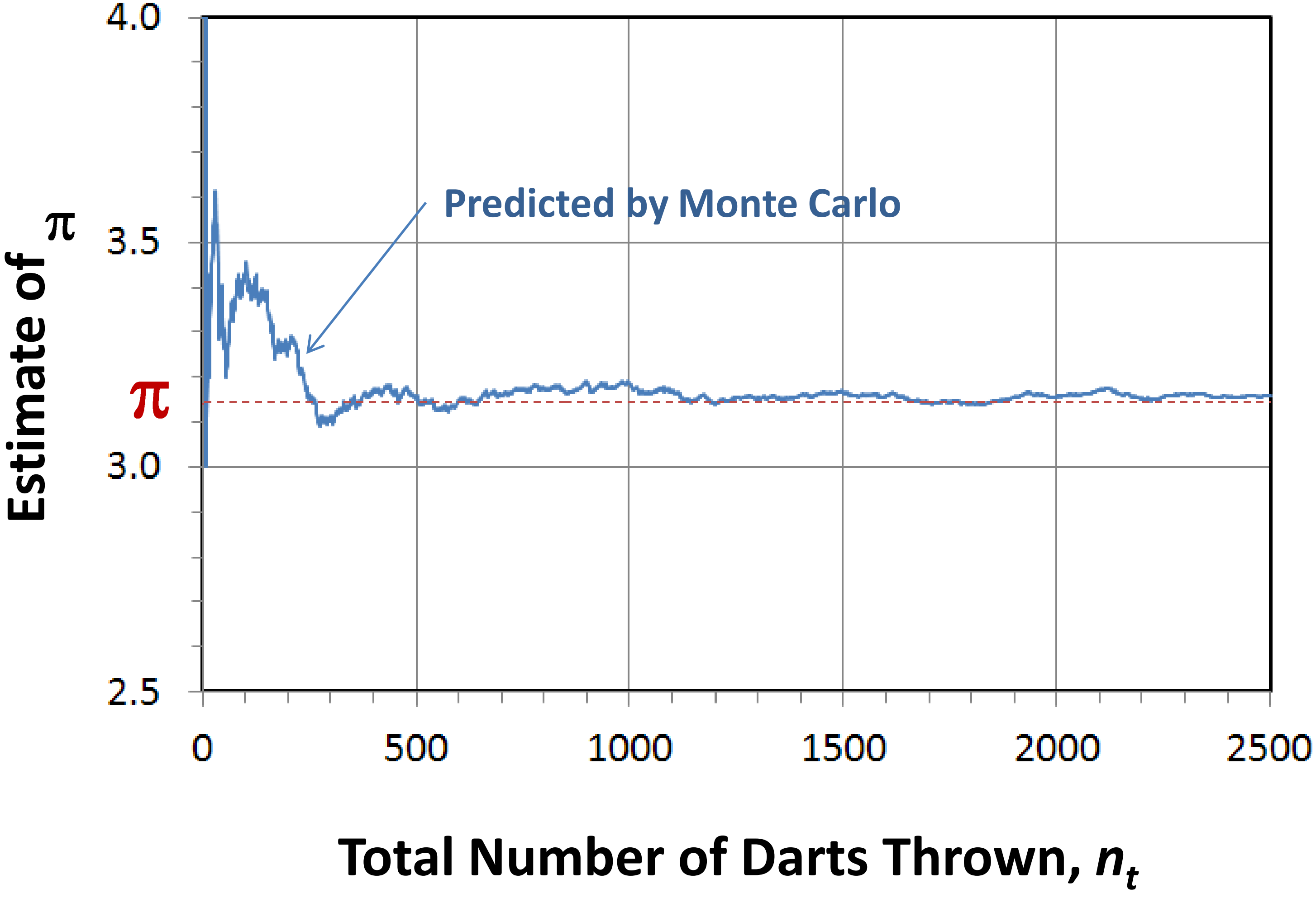
Example -- The "World's Worst Dart Player"

This suggests that our ability to approximate π accurately increases with the total number of darts thrown;

Also, as n_t increases, the addition of a single dart has less of an effect on the approximation of π;

So, instead of simply throwing only a few darts, let's see what happens when we throw *thousands* of darts.

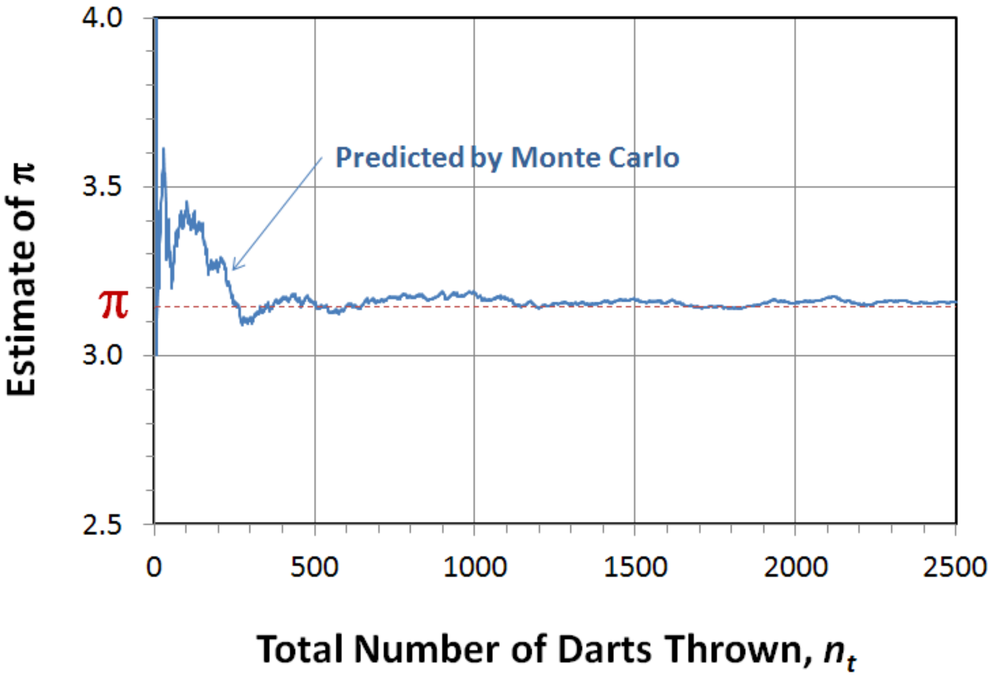
Example -- The "World's Worst Dart Player"



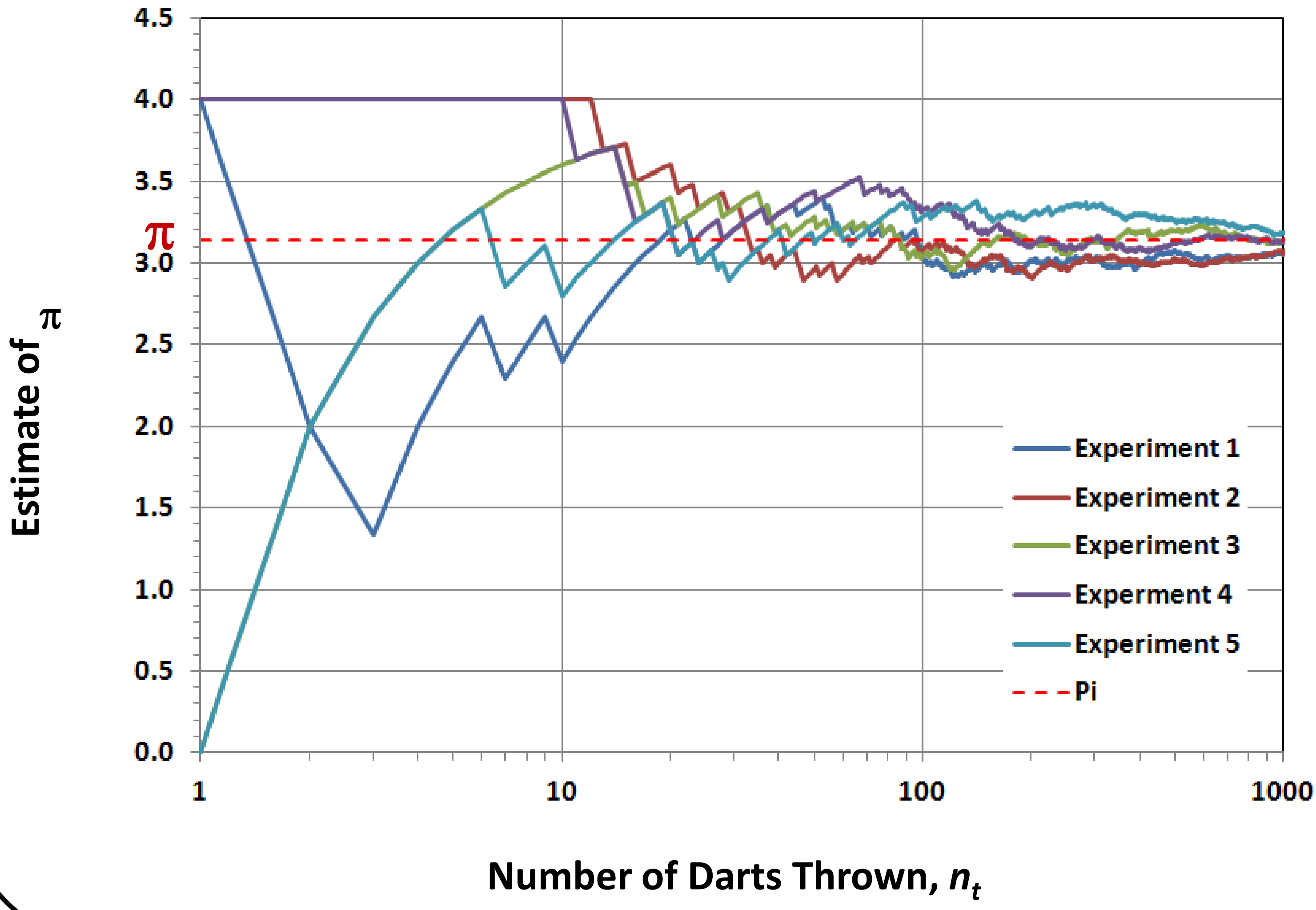
Example -- The "World's Worst Dart Player"

For this example our estimate of π can vary dramatically for only a small number of darts;

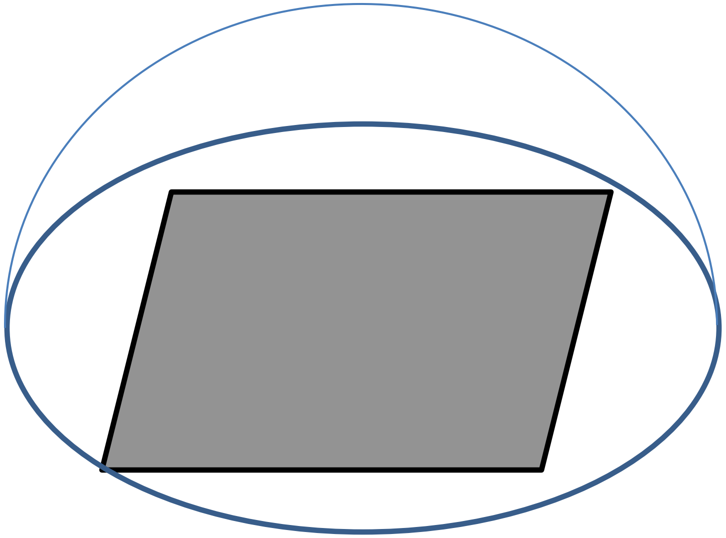
But, eventually, with enough darts, we get an excellent approximation of π.



Example -- The "World's Worst Dart Player"



Calculating Form Factors Using the Monte Carlo Method



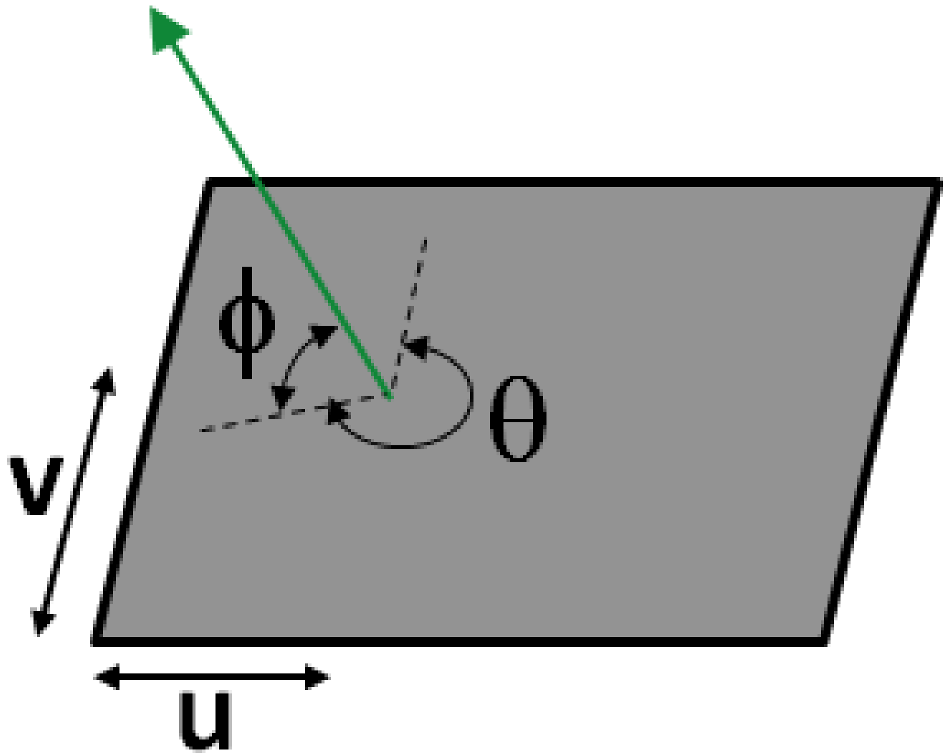
To calculate a form factor, we'd like our Monte Carlo simulation to represent the view from one surface over the region that surface can "see";

For example, a flat plate surface can "see" an entire hemisphere.

Calculating Form Factors Using the Monte Carlo Method

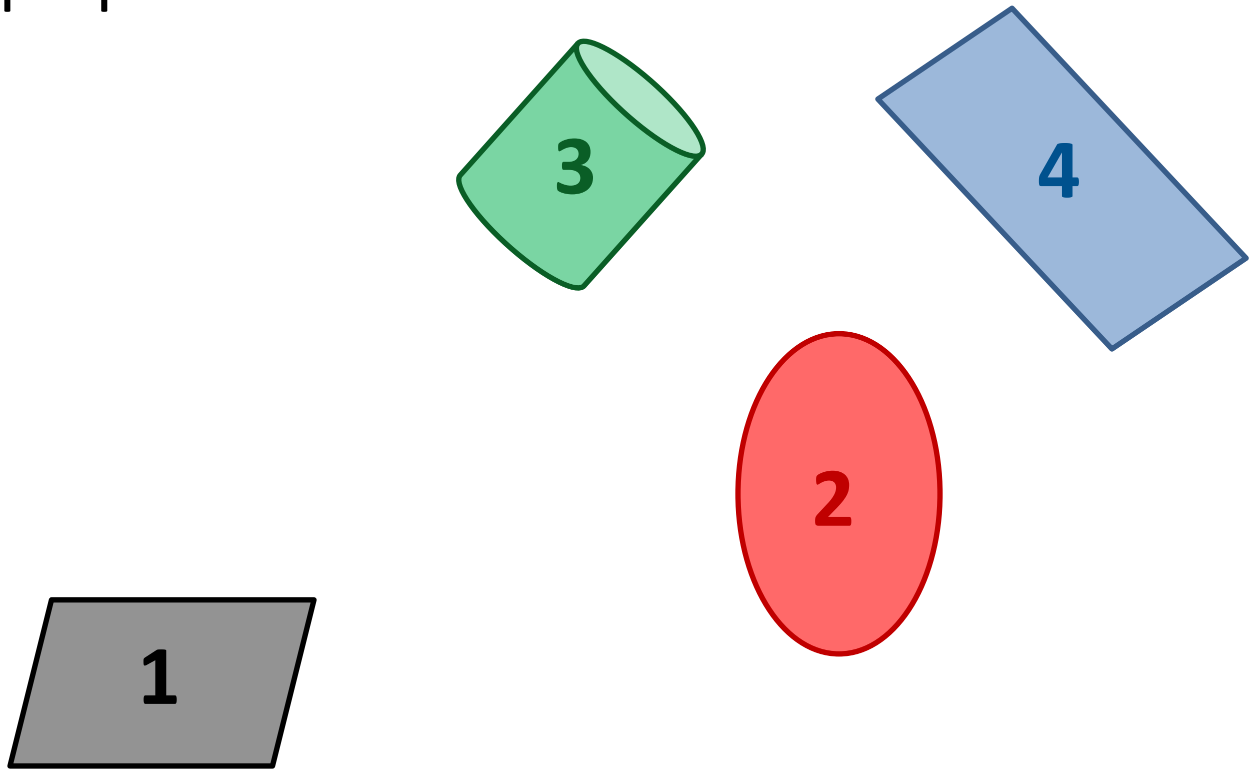
Our integration scheme will need to sample from the entire plate area (*u*, *v*), and all angles (θ, ϕ);

In this case, four random numbers are needed for each ray.



Calculating Form Factors Using the Monte Carlo Method

Consider the following geometry -- all surfaces are opaque.



Calculating Form Factors Using the Monte Carlo Method

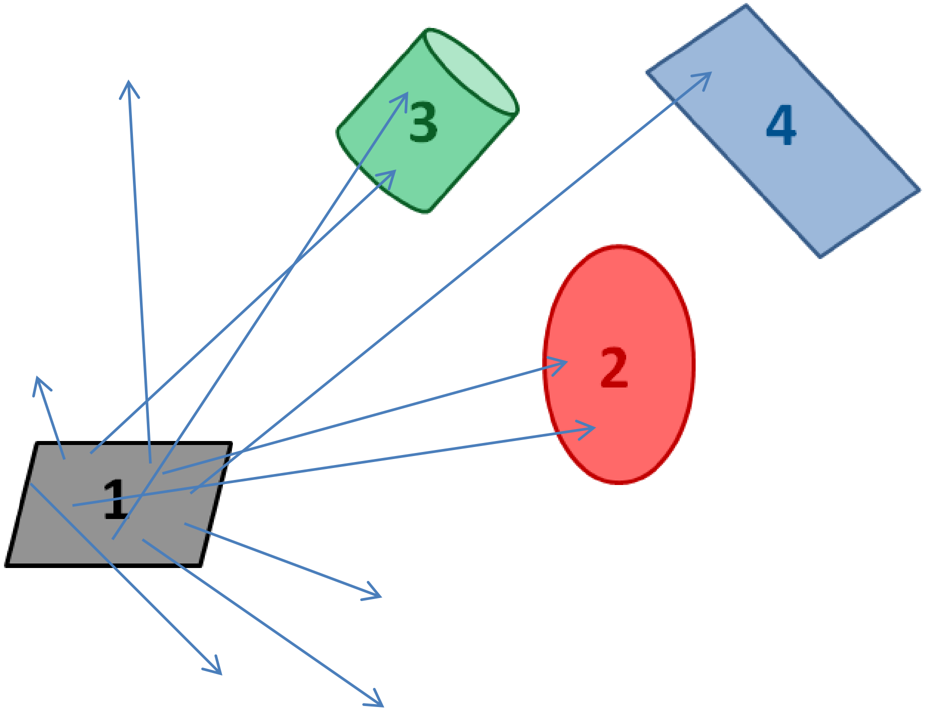
Let's shoot 1000 rays from Surface 1;

200 rays hit Surface 2;

150 rays hit Surface 3;

80 rays hit Surface 4;

The rest escape to Space.



Calculating Form Factors Using the Monte Carlo Method

The form factor between Surface 1 and the others is:

$FF_{12} = 200/1000 = 0.200$

$FF_{13} = 150/1000=0.150$

$FF_{14} = 80/1000 = 0.080$

$FF_{1SPACE} = 570/1000 = 0.570$

Calculating Form Factors Using the Monte Carlo Method

We also see that:

$FF_{12} + FF_{13} + FF_{14} + FF_{1SPACE} =$

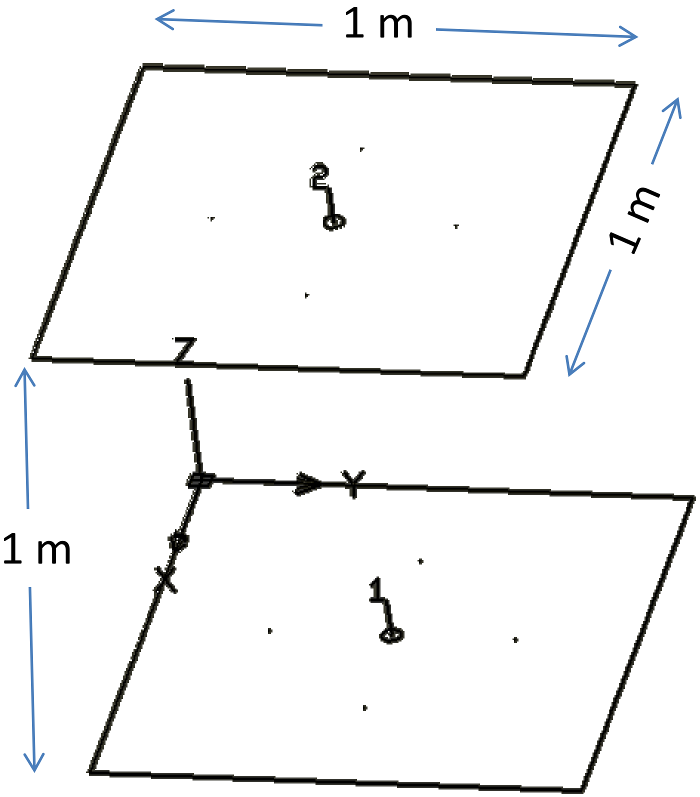
$0.200 + 0.150 + 0.080 + 0.580 = 1.000$

In all cases, form factors must sum to unity.

Sample Problem

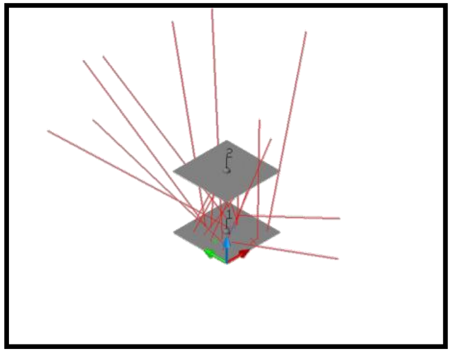
Consider two parallel unit plates separated by one unit;

We'll apply the Monte Carlo method to calculate the form factor between these two surfaces.

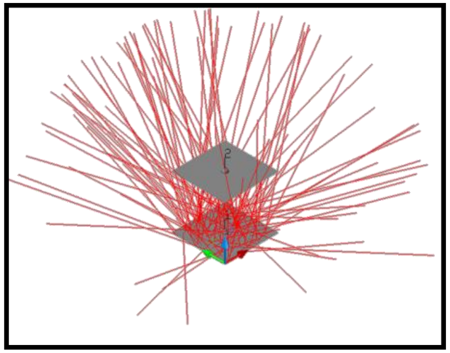


Screen shot from RadCad® by Cullimore and Ring Technologies, Inc

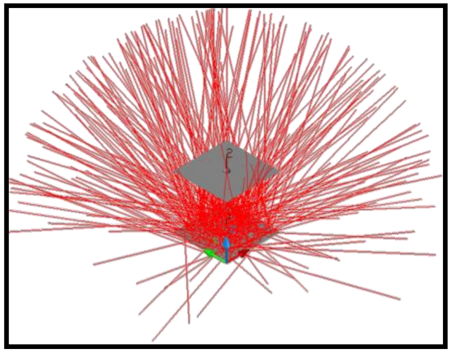
Sample Problem



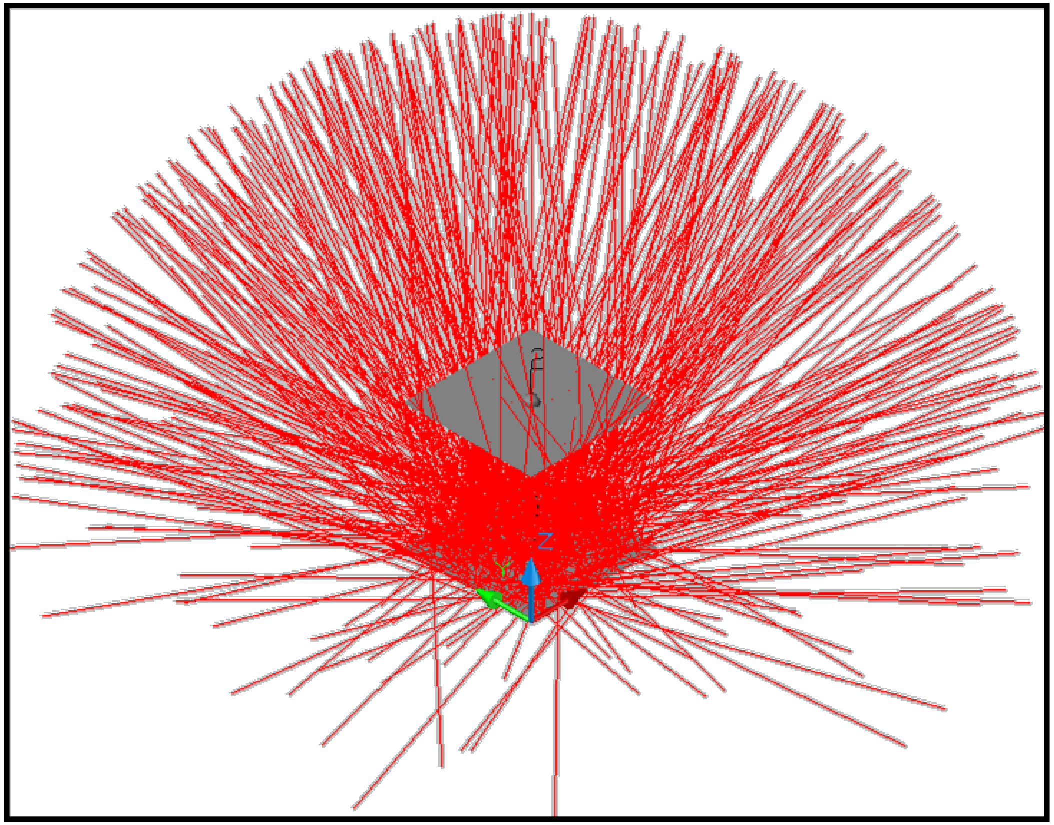
20 rays



220 rays



420 rays

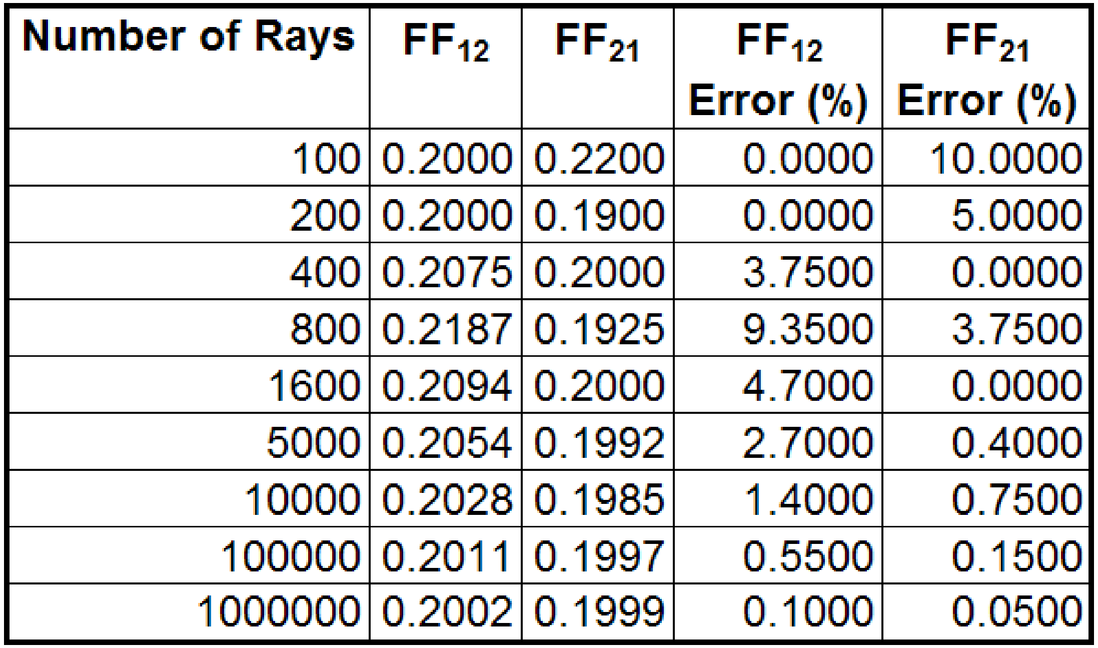


720 rays

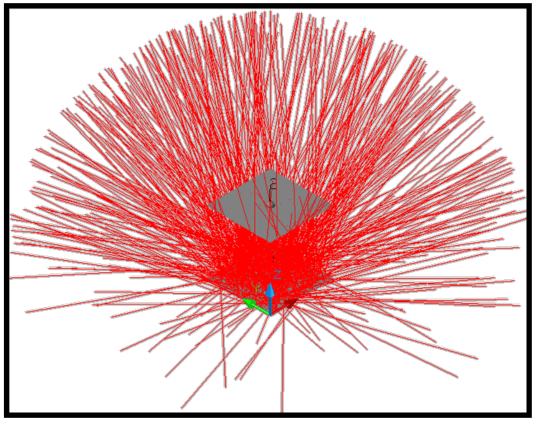
Screen shot from RadCad® by Cullimore and Ring Technologies, Inc

Sample Problem

The solution improves as the number of rays shot increases:



Number of Rays	FF$_{12}$	FF$_{21}$	FF$_{12}$ Error (%)	FF$_{21}$ Error (%)
100	0.2000	0.2200	0.0000	10.0000
200	0.2000	0.1900	0.0000	5.0000
400	0.2075	0.2000	3.7500	0.0000
800	0.2187	0.1925	9.3500	3.7500
1600	0.2094	0.2000	4.7000	0.0000
5000	0.2054	0.1992	2.7000	0.4000
10000	0.2028	0.1985	1.4000	0.7500
100000	0.2011	0.1997	0.5500	0.1500
1000000	0.2002	0.1999	0.1000	0.0500



Screen shot from RadCad® by Cullimore and Ring Technologies, Inc

Statistical Error (Ref. 6)

If we are estimating a value generated by sampling a random distribution, we cannot be 100% certain of a bounding value unless we say that its value lies somewhere between -∞ and + ∞;

So if we're to generate meaningful answers using random variables, we'll need to be more realistic about the bounds;

We can do this by specifying a *confidence interval*.

Statistical Error (Ref. 7)

The Normal or Gaussian Distribution is given by:

$$f(x;\mu,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where...

μ is the mean value, and;
σ is the standard deviation.

Statistical Error (Ref. 7)

We will also consider the Cumulative Density Function (CDF):

$$F(x;\mu,\sigma^2) = \frac{1}{2}\left[1 + erf\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right]$$

The CDF tells us the *probability of a random variable falling in the interval (-∞, x);*

Statistical Error (Ref. 7)

We want to know the probability of a random variable falling in the interval (μ-$N\sigma$, μ+$N\sigma$):

$$\mathrm{F}(\mu + N\sigma; \mu, \sigma^2) - \mathrm{F}(\mu - N\sigma; \mu, \sigma^2) = erf\left(\frac{N}{\sqrt{2}}\right)$$

where *N* is the number of standard deviations (σ) from the mean value (μ).

Statistical Error

For example, for one standard deviation, *N = 1*, the probability of a random variable falling in the interval (μ-σ, μ+σ) is:

$$erf\left(\frac{N}{\sqrt{2}}\right) = erf\left(\frac{1}{\sqrt{2}}\right) = 0.683$$

In other words, we can say that a random variable has a 68.3% chance of being within one standard deviation of the mean -- or, to put it another way, *we're 68.3% confident that the value lies somewhere within one standard deviation of the mean*.

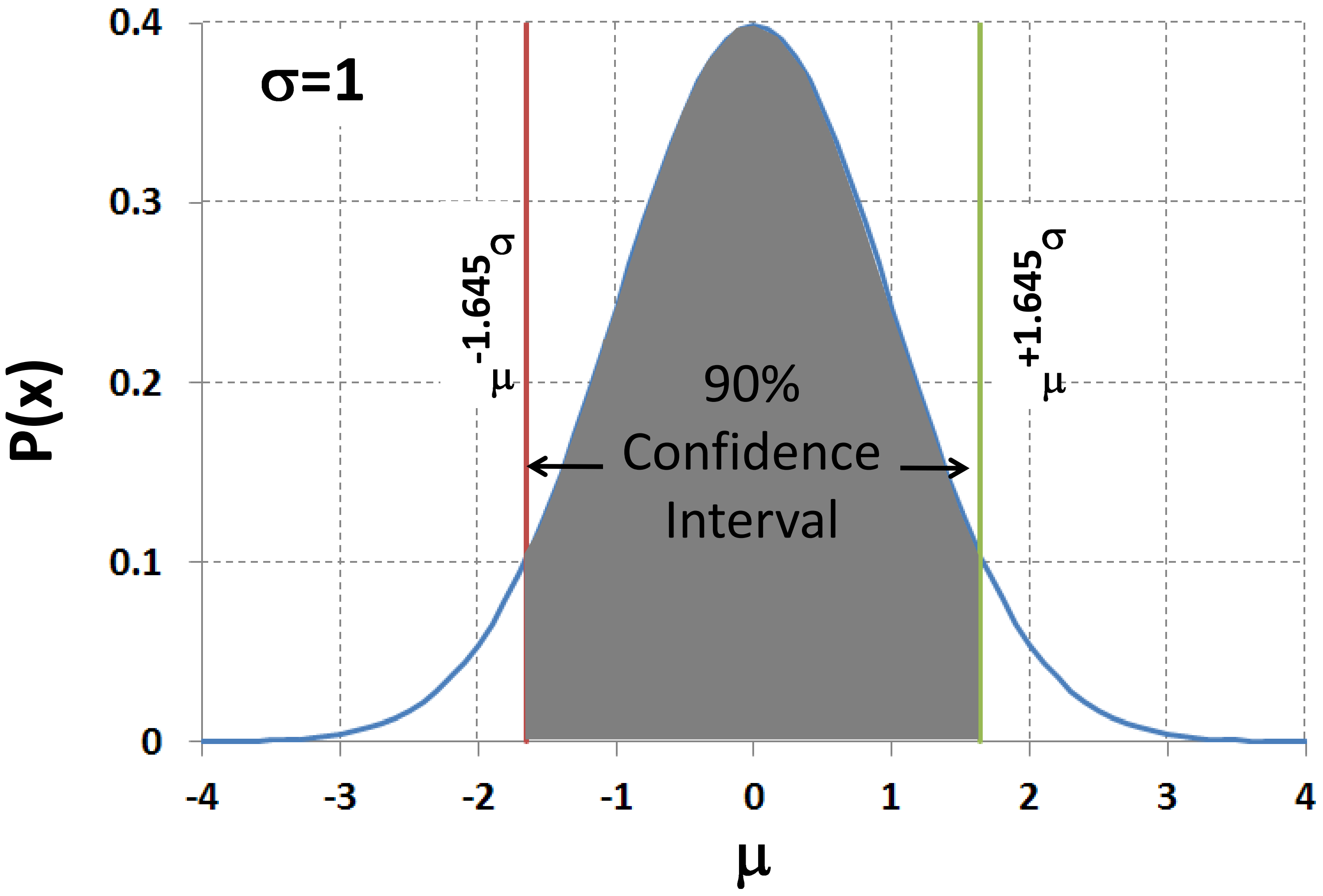
Statistical Error

If we want to establish an answer with 90% confidence, we would need to find *N* for the following:

$$erf\left(\frac{N}{\sqrt{2}}\right) = 0.90$$

This occurs for *N = 1.645*.

Statistical Error



Statistical Error (Ref. 5)

Rays shot for form factor calculations can take on one of two states:

If the ray *hits* a given surface, it *adds 1* to the tally;

If the ray *misses* a given surface, it *adds nothing* to the tally and takes on a value of 0;

For a form factor calculation, the value of the form factor is simply the probability, *p*, that a random ray shot from one surface will strike the other.

Statistical Error (Ref. 6)

With only two possible states, we see this is a discrete distribution;

To calculate the mean, μ of a discrete distribution, we use:

$$\mu = \sum_j x_j f(x_j)$$

where...

x_j is the value of an event, and;
$f(x_j)$ is the probability of the event happening.

Statistical Error (Ref. 6)

For a discrete distribution, the variance, σ^2, is:

$$\sigma^2 = \sum_j (x_j - \mu)^2 f(x_j)$$

Note that *j* = 2 since *x* can take on a value of 0 or 1;

But variance can be applied to, both, of a sample, s^2, or of the mean, σ^2.

Statistical Error (Ref. 6)

Form factor error estimation strategy we assume:

The variance of a sample, s^2, is approximately equal to the variance of the distribution:

$$\sigma^2 \approx s^2$$

The variance of the distribution of means, $\sigma_m{}^2$, calculated from n samples, is estimated by using the variance of the sample population:

$$\sigma_m^2 = \frac{\sigma^2}{n}$$

Statistical Error (Ref. 6)

If the mean value, μ is simply the probability, *p,* of the ray striking the other surface, then the probability of missing the surface is *(1-p)* and the variance becomes:

$$\sigma^2 = \sum_j \left(x_j - \mu\right)^2 f\left(x_j\right)$$

$$= (0 - p)^2(1 - p) + (1 - p)^2 p$$

$$= p^2 - p^3 + p - 2p^2 + p^3$$

$$= p(1 - p)$$

Statistical Error (Ref. 6)

What we really seek is a variance of the means, $\sigma_m{}^2$:

$$\sigma_m^2 = \frac{p(1-p)}{n}$$

For a 90% confidence interval, our estimate of our error:

$$error = 1.645\frac{\sigma_m}{\mu} = 1.645\sqrt{\frac{(1-p)}{np}}$$

Statistical Error (Ref. 6)

$$error = 1.645\sqrt{\frac{(1 - FF)}{nFF}}$$

When *FF* is high, *(1-FF)/FF* is low and *n* doesn't have to be large to achieve low error;

When *FF* is low, *(1-FF)/FF* is high and *n* must be large, relatively speaking, to lower the error.

Statistical Error

How does that change the number of rays ($n_{1/2}$) we have to shoot if we wish to halve the error?

$$0.5 = \frac{error_{1/2}}{error} = \frac{\cancel{1.645}\sqrt{\frac{\cancel{(1-FF)}}{n_{1/2}\cancel{FF}}}}{\cancel{1.645}\sqrt{\frac{\cancel{(1-FF)}}{n\cancel{FF}}}}$$

The number of rays required quadruples: $n_{1/2} = 4n$

Statistical Error (Ref. 8)

For 90% confidence, the relationship between form factor magnitude, number of rays shot and the associated statistical error is:

	Number of Rays							
Form Factor Magnitude	**1K**	**2K**	**5K**	**10K**	**20K**	**50K**	**100K**	**1M**
0.001	164.4	116.3	73.5	52.0	36.8	23.3	16.4	5.2
0.002	116.2	82.2	52.0	36.7	26.0	16.4	11.6	3.7
0.005	73.4	51.9	32.8	23.2	16.4	10.4	7.3	2.3
0.01	51.8	36.6	23.1	16.4	11.6	7.3	5.2	1.6
0.02	36.4	25.7	16.3	11.5	8.1	5.1	3.6	1.2
0.05	22.7	16.0	10.1	7.2	5.1	3.2	2.3	0.7
0.1	15.6	11.0	7.0	4.9	3.5	2.2	1.6	0.5
0.2	10.4	7.4	4.7	3.3	2.3	1.5	1.0	0.3
0.5	5.2	3.7	2.3	1.6	1.2	0.7	0.5	0.2

% Error

Form Factor Reciprocity and Combinations

Reciprocity

For two nodes *i* and *j*, we observe the following relationship:

$$A_i FF_{ij} = A_j FF_{ji}$$

This is called reciprocity;

We can use this to solve for unknown form factors in our geometry.

Form Factor Combinations (Ref. 3)

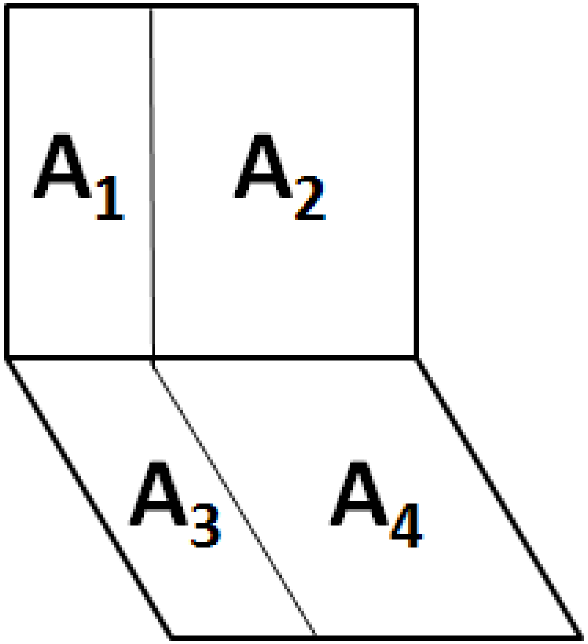
Reciprocity, in combination with form factor algebra, can be used to identify the following relationships:

$$A_1 FF_{1-3} = A_3 FF_{3-1}$$

$$A_1 FF_{1-34} = A_1 FF_{1-3} + A_1 FF_{1-4}$$

$$A_{12} FF_{12-34} = A_1 FF_{1-34} + A_2 F_{2-34}$$

$$A_{12} FF_{12-34} = A_1 FF_{1-3} + A_1 FF_{1-4} + A_2 FF_{2-3} + A_2 FF_{2-4}$$



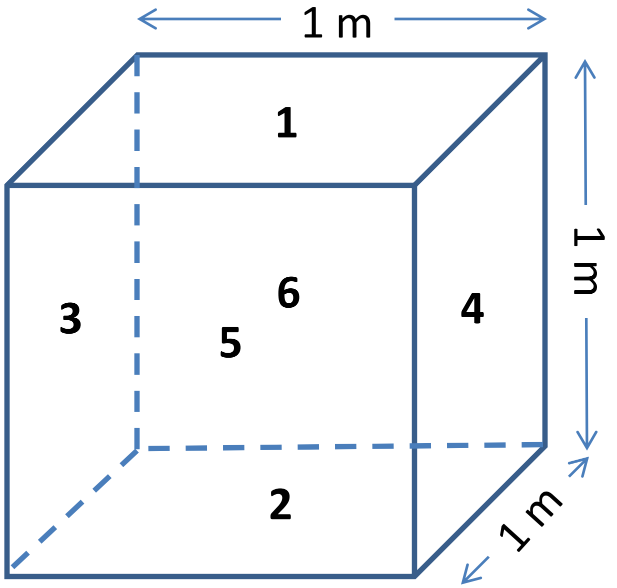
Form Factor Combinations

Let's look at the form factor relationships for a cube:

$$\sum_{j=2}^{6} FF_{1j} = 1.0$$

$$FF_{1-2} = 0.2$$

$$FF_{1-3456} = 1.0 - 0.2 = 0.8$$



Form Factor Combinations

We see that:

$$A_1 FF_{1-3456} = A_1 F_{1-3} + A_1 FF_{1-4} + A_1 FF_{1-5} + A_1 FF_{1-6}$$

But we see symmetry in this cube geometry so we conclude that:

$$FF_{1-3} = FF_{1-4} = FF_{1-5} = FF_{1-6} = 0.2$$

We'll use this result in an example problem later.

Grey Body Factors

Difference Between a Form Factor and Grey Body Factor

A **form factor** describes how well one object "sees" another object via direct view;

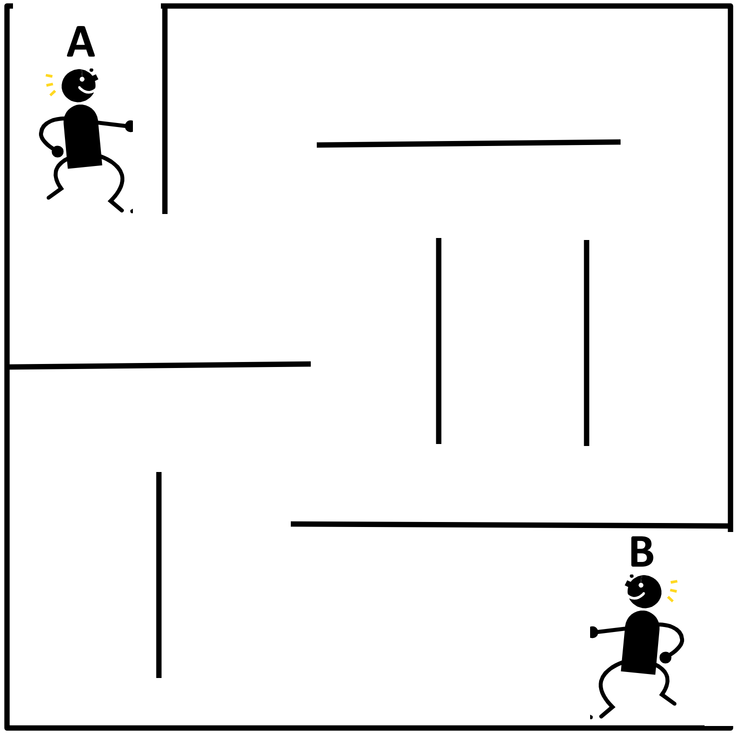
In other words, a **form factor** *quantifies the fraction of energy emitted from one surface that arrives at another surface directly*;

A **grey body factor** *quantifies the fraction of energy leaving one surface that is absorbed another surface through all possible paths*.

The Mirror Maze Analogy

Consider two friends, **A** and **B** situated at opposite ends of a maze comprised of mirrors;

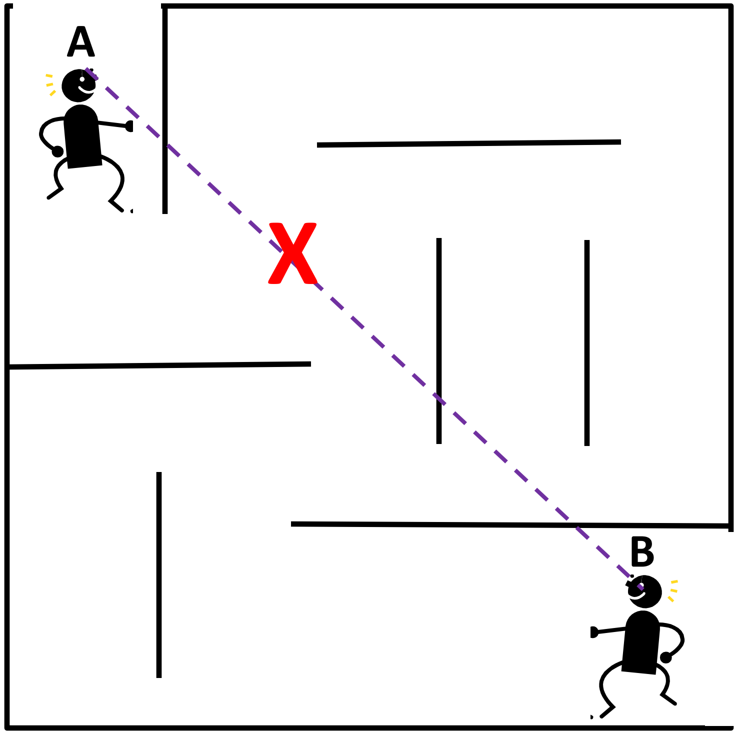
We ask, can the two friends see one another?



The Mirror Maze Analogy

A cannot see **B** *directly*;

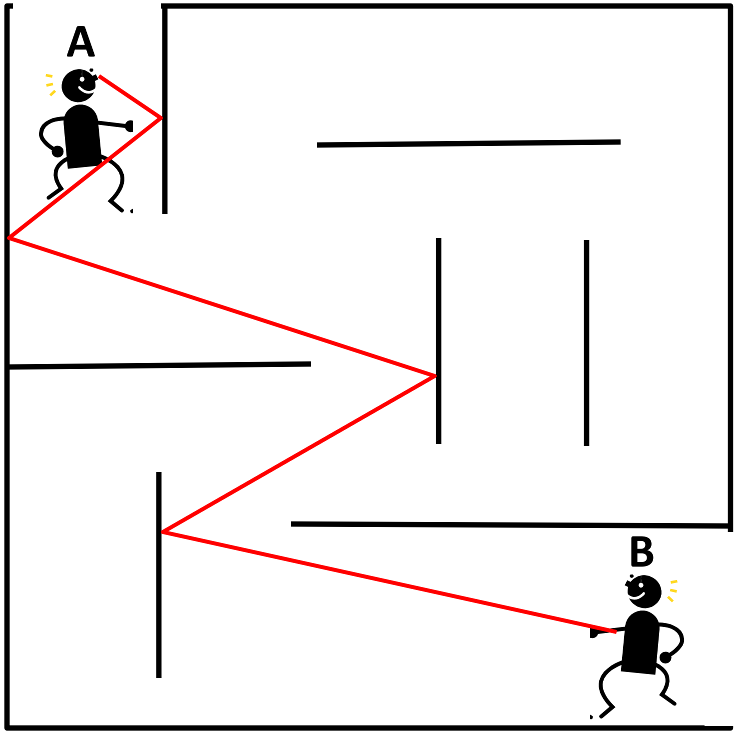
We say that *the form factor between **A** and **B** is zero*;



The Mirror Maze Analogy

But if the walls are mirrors, **A** *can see* **B** via reflection;

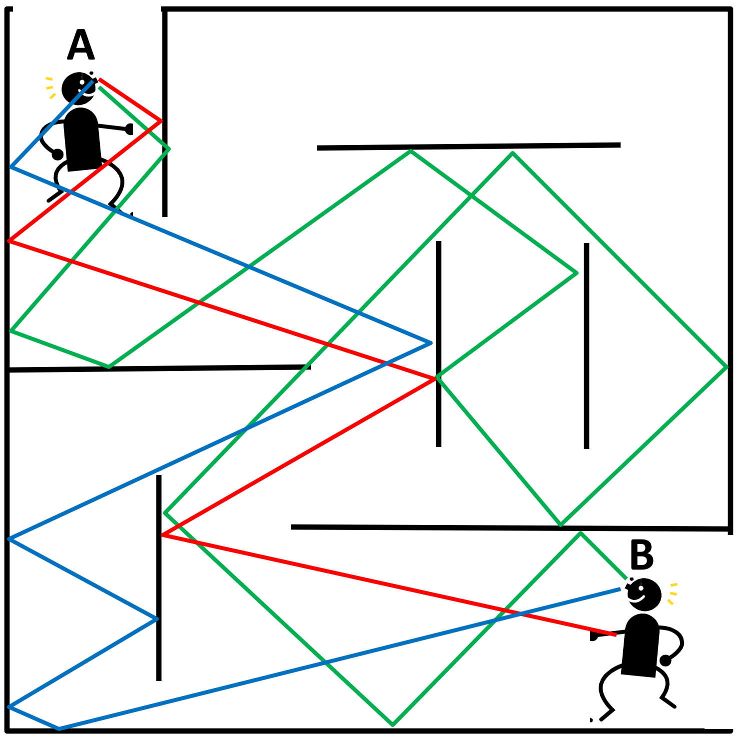
In this case, the grey body factor between **A** and **B** is greater than zero.



The Mirror Maze Analogy

Actually, there are many possible reflections that will allow **A** to see **B**;

The grey body factor is the amount of energy that leaves **A** and is absorbed at **B** *via all possible paths*.



Gebhart's Method

Turning Form Factors Into Grey Body Factors

Form factors are an intermediate product and, by themselves, have limited use;

We really seek a means of determining how radiant energy is distributed in a thermal network;

Form factors and surface emittances are inputs to Gebhart's method to calculate grey body factors which can readily be turned into Radks.

This method works for diffuse interchange only.

Gebhart's Method (from Ref. 9)

Define the Gebhart factor as:

$$B_{ij} = \frac{Energy\ absorbed\ at\ A_j originating\ as\ emission\ at\ A_i}{Total\ radiation\ emitted\ at\ A_i}$$

where *i* is one surface and *j* is another and A_i and A_j are surface areas of *i* and *j*, respectively.

Gebhart's Method (from Ref. 9)

For N_s objects, the general formula is:

$$B_{ij} = \varepsilon_j FF_{ij} + \sum_{k=1}^{N_s} \left((1 - \varepsilon_k) FF_{ik} B_{kj} \right)$$

We also note for an object, *i*:

$$\sum_{j=1}^{N_s} B_{ij} = 1$$

Gebhart's Method

For three objects that see *only* one another, we can establish the B_{ij} values, holding i constant:

$$B_{11} = \varepsilon_1 FF_{11} + (1-\varepsilon_1)FF_{11}B_{11} + (1-\varepsilon_2)FF_{12}B_{21} + (1-\varepsilon_3)FF_{13}B_{31}$$

$$B_{12} = \varepsilon_1 FF_{12} + (1-\varepsilon_1)FF_{21}B_{11} + (1-\varepsilon_2)FF_{22}B_{21} + (1-\varepsilon_3)FF_{23}B_{31}$$

$$B_{13} = \varepsilon_1 FF_{13} + (1-\varepsilon_1)FF_{31}B_{11} + (1-\varepsilon_2)FF_{32}B_{21} + (1-\varepsilon_3)FF_{33}B_{31}$$

By remembering that $B_{11} + B_{12} + B_{13} = 1$ and noting that reflectance, $\rho = (1-\varepsilon)$, we can interpret each of these terms.

Gebhart's Method (Ref. 10)

$$\begin{matrix} B_{11} = \\ B_{12} = \\ B_{13} = \end{matrix} \boxed{\begin{matrix} \varepsilon_1 FF_{11} \\ \varepsilon_1 FF_{12} \\ \varepsilon_1 FF_{13} \end{matrix}} \begin{matrix} + \\ + \\ + \end{matrix} \boxed{\begin{matrix} (1-\varepsilon_1)FF_{11}B_{11} \\ (1-\varepsilon_1)FF_{11}B_{12} \\ (1-\varepsilon_1)FF_{11}B_{13} \end{matrix}} \begin{matrix} + \\ + \\ + \end{matrix} \boxed{\begin{matrix} (1-\varepsilon_2)FF_{12}B_{21} \\ (1-\varepsilon_2)FF_{12}B_{22} \\ (1-\varepsilon_2)FF_{12}B_{23} \end{matrix}} \begin{matrix} + \\ + \\ + \end{matrix} \boxed{\begin{matrix} (1-\varepsilon_3)FF_{13}B_{31} \\ (1-\varepsilon_3)FF_{13}B_{32} \\ (1-\varepsilon_3)FF_{13}B_{33} \end{matrix}}$$

Fraction leaving object 1...
directly absorbed at j=1, 2, 3;
reflecting off of 1 and is absorbed j=1, 2, 3;
reflecting off of 2 and is absorbed at j=1, 2, 3;
reflecting off of 3 and is absorbed at j=1, 2, 3;

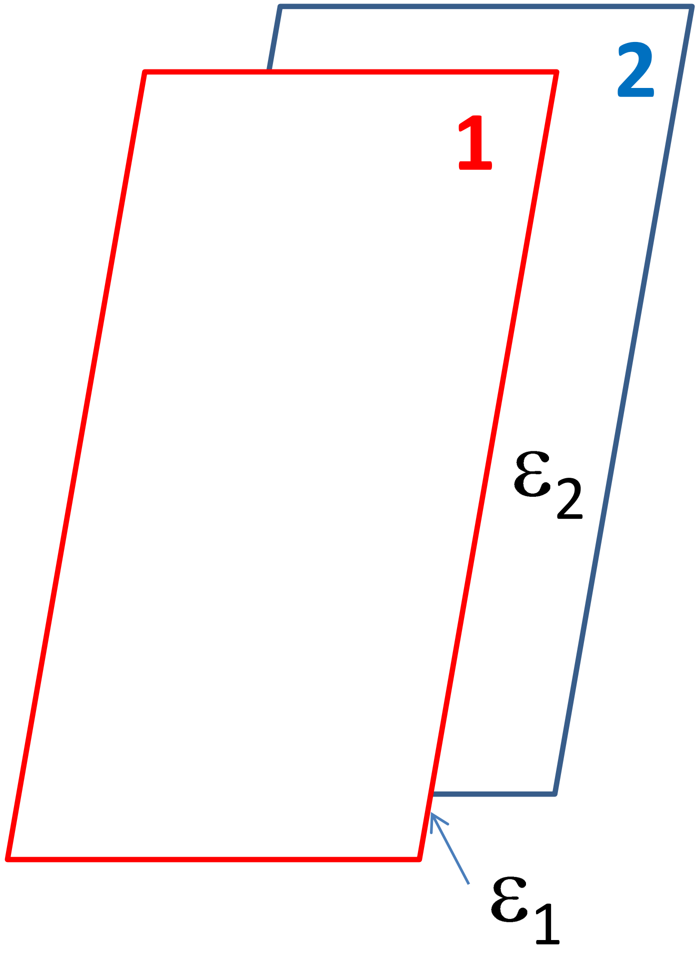
The B_{1j} for $j = 1, 2, 3$ must sum to unity.

Gebhart's Method

Consider two infinite parallel plates with the specified properties;

Given the infinite extent, we can say:

$$FF_{12} = FF_{21} = 1$$
$$FF_{11} = 0$$
$$FF_{22} = 0$$



Gebhart's Method

Our expression for the B_{ij} values becomes:

$$B_{11} = \varepsilon_1 FF_{11} + (1 - \varepsilon_1) FF_{11} B_{11} + (1 - \varepsilon_2) FF_{12} B_{21}$$

$$B_{21} = \varepsilon_1 FF_{21} + (1 - \varepsilon_1) FF_{21} B_{11} + (1 - \varepsilon_2) FF_{22} B_{21}$$

Gebhart's Method

Rearranging, yields the following:

$$B_{11} - (1 - \varepsilon_1)FF_{11}B_{11} - (1 - \varepsilon_2)FF_{12}B_{21} = \varepsilon_1 FF_{11}$$

$$B_{21} - (1 - \varepsilon_1)FF_{21}B_{11} - (1 - \varepsilon_2)FF_{22}B_{21} = \varepsilon_1 FF_{21}$$

Grouping similar terms together:

$$[1 - (1 - \varepsilon_1)FF_{11}]B_{11} - [(1 - \varepsilon_2)FF_{12}]B_{21} = \varepsilon_1 FF_{11}$$

$$-[(1 - \varepsilon_1)FF_{21}]B_{11} + [1 - (1 - \varepsilon_2)FF_{22}]B_{21} = \varepsilon_1 FF_{21}$$

Gebhart's Method

In matrix form, the system of equations becomes:

$$\begin{bmatrix} [1-(1-\varepsilon_1)FF_{11}] & -[(1-\varepsilon_2)FF_{12}] \\ -[(1-\varepsilon_1)FF_{21}] & [1-(1-\varepsilon_2)FF_{22}] \end{bmatrix} \begin{Bmatrix} B_{11} \\ B_{21} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_1 FF_{11} \\ \varepsilon_1 FF_{21} \end{Bmatrix}$$

Recall, for infinite parallel plates:

$$FF_{12} = FF_{12} = 1$$
$$FF_{11} = 0$$
$$FF_{22} = 0$$

Gebhart's Method

Our system of equations becomes:

$$\begin{bmatrix} 1 & -(1-\varepsilon_2) \\ -(1-\varepsilon_1) & 1 \end{bmatrix} \begin{Bmatrix} B_{11} \\ B_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \varepsilon_1 \end{Bmatrix}$$

Gebhart's Method

From the first equation, we see that:

$$B_{11} = (1 - \varepsilon_2)B_{21}$$

Substituting into the second equation yields:

$$-(1 - \varepsilon_1)(1 - \varepsilon_2)B_{21} + B_{21} = \varepsilon_1$$

Finally, solving for B_{21}:

$$B_{21} = \frac{\varepsilon_1}{\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} - 1}$$

Gebhart's Method

You may be more familiar with the infinite parallel plate solution that looks like this:

$$\mathcal{F}_{21} = \frac{1}{\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} - 1}$$

$\mathcal{F}_{21}$ is the ratio of the energy <u>incident</u> at surface ***1*** originating at surface ***2*** divided by the total radiation emitted at ***2.***

Gebhart's Method

But, remember, the B_{21} is the ratio of the energy absorbed at surface ***1*** originating as emission at surface ***2*** divided by the total radiation emitted at ***2***;

$$B_{21} = \frac{\varepsilon_1}{\dfrac{1}{\varepsilon_1} + \dfrac{1}{\varepsilon_2} - 1}$$

The relationship between the two is:

$$B_{21} = \varepsilon_1 \mathcal{F}_{21}$$

Approximating $\mathscr{F}_{21}$(Ref. 11)

We can establish an approximate value of $\mathscr{F}_{21}$ using a simpler expression when the surface emittance values are relatively high.

To do this, we can modify our expression for $\mathscr{F}_{21}$ by multiplying the numerator and denominator by $\varepsilon_1\varepsilon_2$:

$$\mathcal{F}_{21} = \frac{1}{\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} - 1} = \left(\frac{\varepsilon_1 \varepsilon_2}{\varepsilon_1 \varepsilon_2}\right) \frac{1}{\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} - 1}$$

Approximating $\mathscr{F}_{21}$(Ref. 11)

This simplifies to:

$$\left(\frac{\varepsilon_1\varepsilon_2}{\varepsilon_1\varepsilon_2}\right)\frac{1}{\dfrac{1}{\varepsilon_1}+\dfrac{1}{\varepsilon_2}-1}=\frac{\varepsilon_1\varepsilon_2}{\varepsilon_2+\varepsilon_1-\varepsilon_1\varepsilon_2}$$

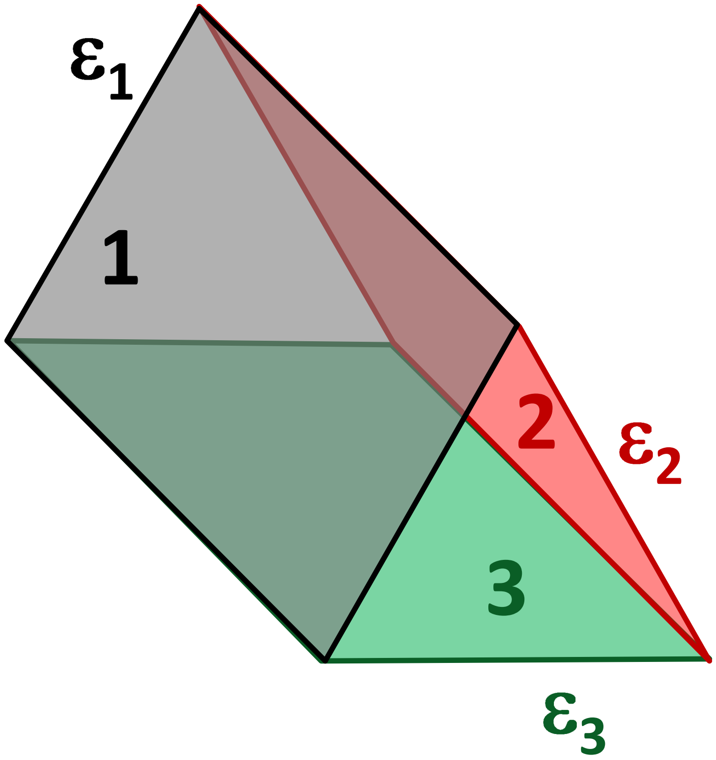
As ε_1 and ε_2 approach unity, the expression approaches $\varepsilon_1 \times \varepsilon_2$.

Approximating $\mathscr{F}_{21}$(Ref. 11)

ε_1	ε_2	$\mathscr{F}_{12}$	Approximate $\varepsilon_1 \times \varepsilon_2$	Approx./$\mathscr{F}_{12}$	Error (%)
0.05	0.05	0.026	0.003	0.098	90.3
0.10	0.10	0.053	0.010	0.190	81.0
0.15	0.15	0.081	0.023	0.278	72.3
0.20	0.20	0.111	0.040	0.360	64.0
0.25	0.25	0.143	0.063	0.438	56.3
0.30	0.30	0.176	0.090	0.510	49.0
0.35	0.35	0.212	0.123	0.578	42.3
0.40	0.40	0.250	0.160	0.640	36.0
0.45	0.45	0.290	0.203	0.698	30.3
0.50	0.50	0.333	0.250	0.750	25.0
0.55	0.55	0.379	0.303	0.798	20.3
0.60	0.60	0.429	0.360	0.840	16.0
0.65	0.65	0.481	0.423	0.878	12.3
0.70	0.70	0.538	0.490	0.910	9.0
0.75	0.75	0.600	0.563	0.938	6.3
0.80	0.80	0.667	0.640	0.960	4.0
0.85	0.85	0.739	0.723	0.978	2.3
0.90	0.90	0.818	0.810	0.990	1.0
0.95	0.95	0.905	0.903	0.998	0.3
1.00	1.00	1.000	1.000	1.000	0.0

Gebhart's Method Example

Consider an infinitely long enclosure with an equilateral triangular cross section;



We wish to calculate the radiation interchange for the internal surfaces given:

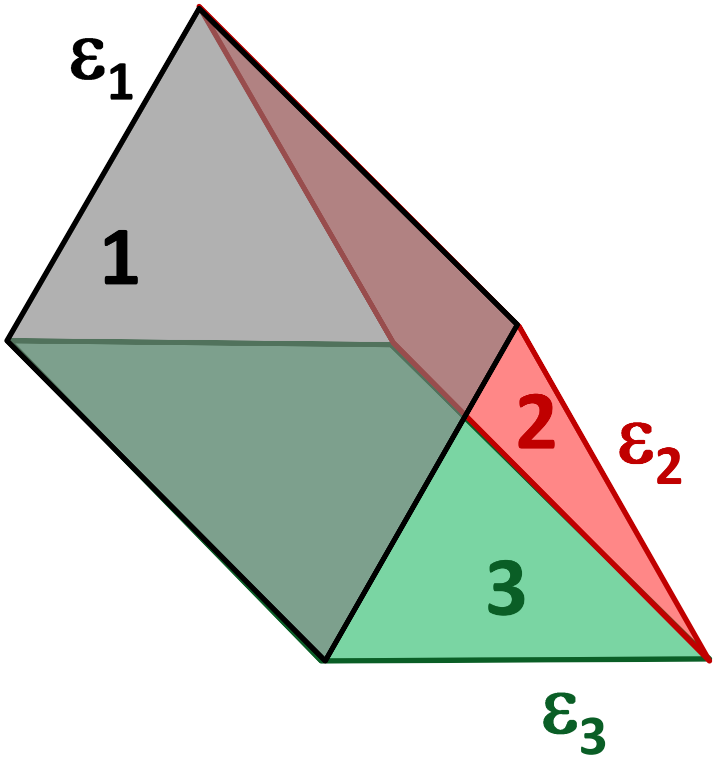
$\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 0.05$

Gebhart's Method Example

Due to infinite extent and problem symmetry:

$FF_{11} = FF_{22} = FF_{33} = 0$
$FF_{12} = FF_{13} = 0.5$
$FF_{21} = FF_{23} = 0.5$
$FF_{31} = FF_{32} = 0.5$

The *FF* sum for each node must equal unity. All *FF* to Space equal zero.



Gebhart's Method Example

We apply our equation for B_{ij} where N_s = 3:

$$B_{ij} = \varepsilon_j FF_{ij} + \sum_{k=1}^{N_S} \left((1 - \varepsilon_k) FF_{ik} B_{kj} \right)$$

Our system of equations is:

$$B_{11} = \varepsilon_1 FF_{11} + (1 - \varepsilon_1) FF_{11} B_{11} + (1 - \varepsilon_2) FF_{12} B_{21} + (1 - \varepsilon_3) FF_{13} B_{31}$$
$$B_{21} = \varepsilon_1 FF_{21} + (1 - \varepsilon_1) FF_{21} B_{11} + (1 - \varepsilon_2) FF_{22} B_{21} + (1 - \varepsilon_3) FF_{23} B_{31}$$
$$B_{31} = \varepsilon_1 FF_{31} + (1 - \varepsilon_1) FF_{31} B_{11} + (1 - \varepsilon_2) FF_{32} B_{21} + (1 - \varepsilon_3) FF_{33} B_{31}$$

Gebhart's Method Example

Rearranging yields:

$$B_{11} - (1 - \varepsilon_1)FF_{11}B_{11} - (1 - \varepsilon_2)FF_{12}B_{21} - (1 - \varepsilon_3)FF_{13}B_{31} = \varepsilon_1 FF_{11}$$

$$B_{21} - (1 - \varepsilon_1)FF_{21}B_{11} - (1 - \varepsilon_2)FF_{22}B_{21} - (1 - \varepsilon_3)FF_{23}B_{31} = \varepsilon_1 FF_{21}$$

$$B_{31} - (1 - \varepsilon_1)FF_{31}B_{11} - (1 - \varepsilon_2)FF_{32}B_{21} - (1 - \varepsilon_3)FF_{33}B_{31} = \varepsilon_1 FF_{31}$$

In matrix form, the system of equations becomes:

$$\begin{bmatrix} 1 - (1 - \varepsilon_1)FF_{11} & -(1 - \varepsilon_2)FF_{12} & -(1 - \varepsilon_3)FF_{13} \\ -(1 - \varepsilon_1)FF_{21} & 1 - (1 - \varepsilon_2)FF_{22} & -(1 - \varepsilon_3)FF_{23} \\ -(1 - \varepsilon_1)FF_{31} & -(1 - \varepsilon_2)FF_{32} & 1 - (1 - \varepsilon_3)FF_{33} \end{bmatrix} \begin{Bmatrix} B_{11} \\ B_{21} \\ B_{31} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_1 FF_{11} \\ \varepsilon_1 FF_{21} \\ \varepsilon_1 FF_{31} \end{Bmatrix}$$

Gebhart's Method Example

Substituting values into the matrices:

$$\begin{bmatrix} 1 & -0.475 & -0.475 \\ -0.475 & 1 & -0.475 \\ -0.475 & -0.475 & 1 \end{bmatrix} \begin{Bmatrix} B_{11} \\ B_{21} \\ B_{31} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0.025 \\ 0.025 \end{Bmatrix}$$

Solving for B_{11}, B_{21} and B_{31}:

$$\begin{Bmatrix} B_{11} \\ B_{21} \\ B_{31} \end{Bmatrix} = \begin{Bmatrix} 0.32203 \\ 0.33898 \\ 0.33898 \end{Bmatrix}$$

Imperfect Form Factor Sums

We can use this example to explore the effect of imperfect form factor sums on energy conservation;

Consider the same geometry but assume, this time, that whichever form factor approximation methodology we used left us with imperfect form factor sums:

$$FF_{12} = FF_{23} = FF_{13} = 0.495$$

with other *FF*s inferred through symmetry.

Imperfect Form Factor Sums

In this instance, our form factors do not sum to unity:

$$\sum_{j=1}^{3} FF_{1j} = 0.0 + 0.495 + 0.495 = 0.990$$

How does this affect energy conservation in the radiation network?

Imperfect Form Factor Sums

If we substitute the imperfect form factors into our system of equations, we get:

$$\begin{bmatrix} 1 & -0.47025 & -0.47025 \\ -0.47025 & 1 & -0.47025 \\ -0.47025 & -0.47025 & 1 \end{bmatrix} \begin{Bmatrix} B_{11} \\ B_{21} \\ B_{31} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0.02475 \\ 0.02475 \end{Bmatrix}$$

Solving once again for B_{11}, B_{21} and B_{31}:

$$\begin{Bmatrix} B_{11} \\ B_{21} \\ B_{31} \end{Bmatrix} = \begin{Bmatrix} 0.26609 \\ 0.28292 \\ 0.28292 \end{Bmatrix}$$

Imperfect Form Factor Sums

A check of energy conservation yields an imperfect summation:

$$B_{11} + B_{12} + B_{13} = 0.26609 + 0.28292 + 0.28292 = 0.83193$$

In this example, only about 83% of the thermal radiation interchange is represented by the network;

This result cautions the analyst to ensure form factors sum to unity for each node.

Calculating Grey Body Factors Using the Monte Carlo Ray Tracing Method

Calculating Grey Body Factors Using the Monte Carlo Method

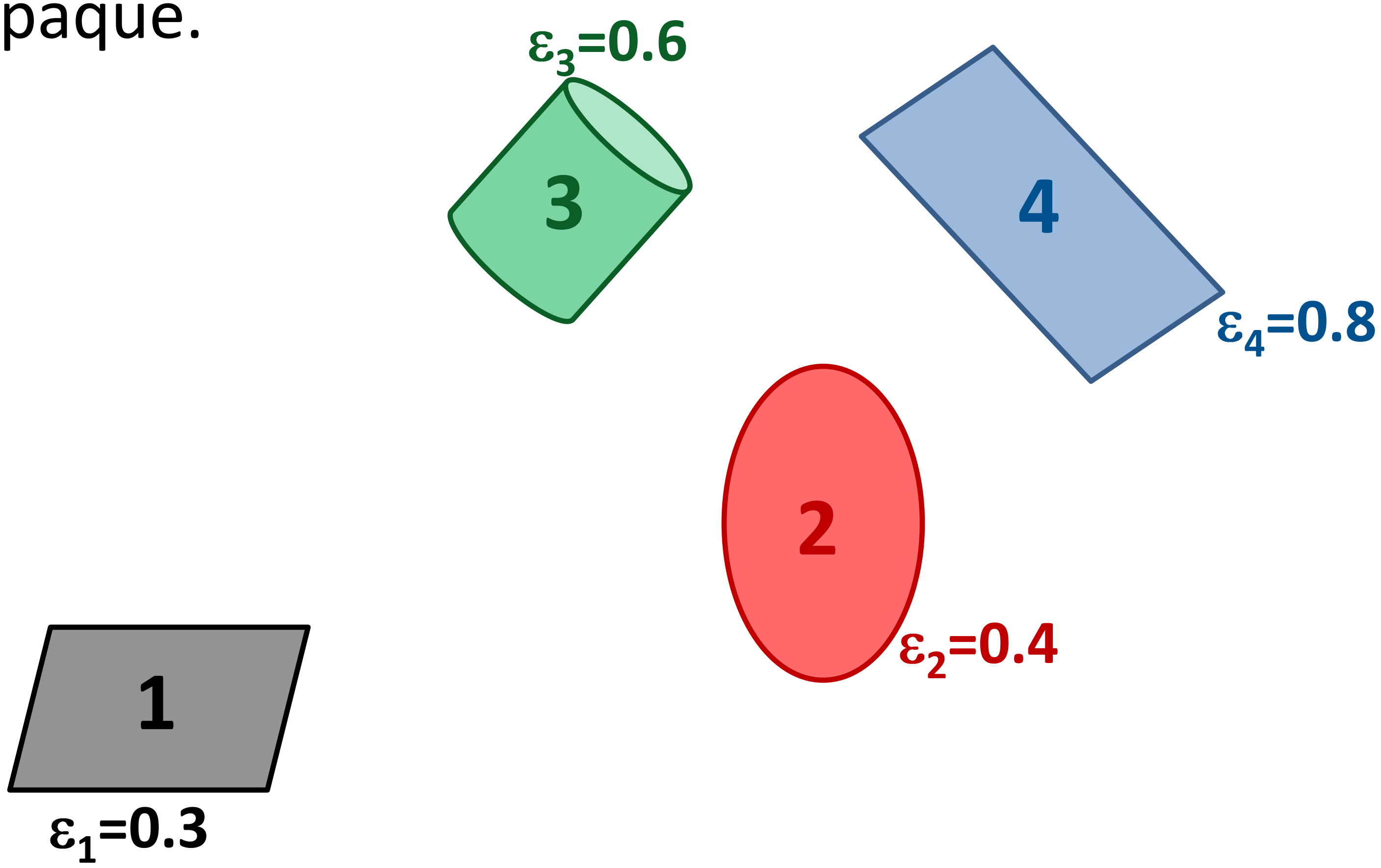
We saw that Monte Carlo ray tracing can be used to calculate form factors;

By extension, we can also use them to calculate grey body factors;

Let's see how this works.

Calculating Grey Body Factors Using the Monte Carlo Method

Consider the following geometry -- all surfaces are opaque.

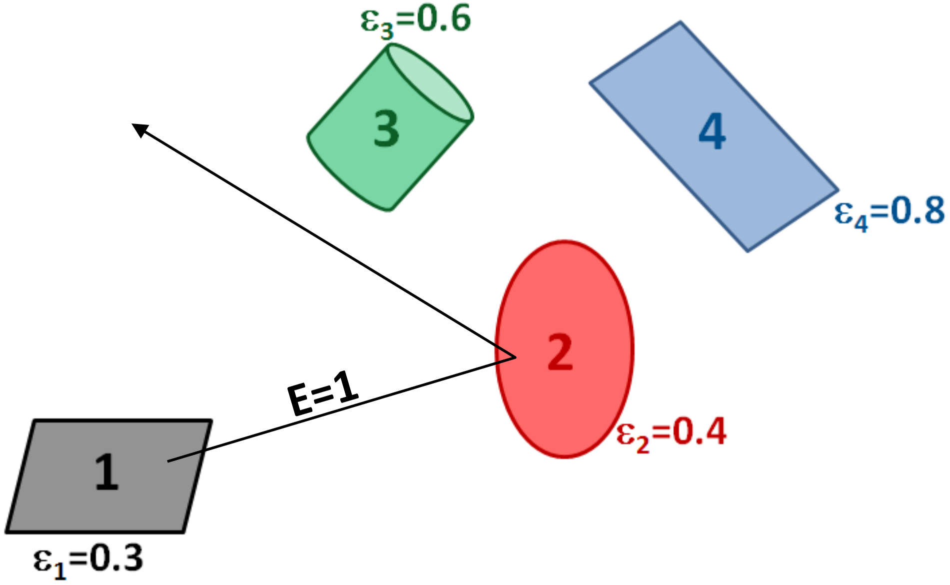


Calculating Grey Body Factors Using the Monte Carlo Method

A ray with unit energy is fired from a random location on Surface 1 and in a random direction;

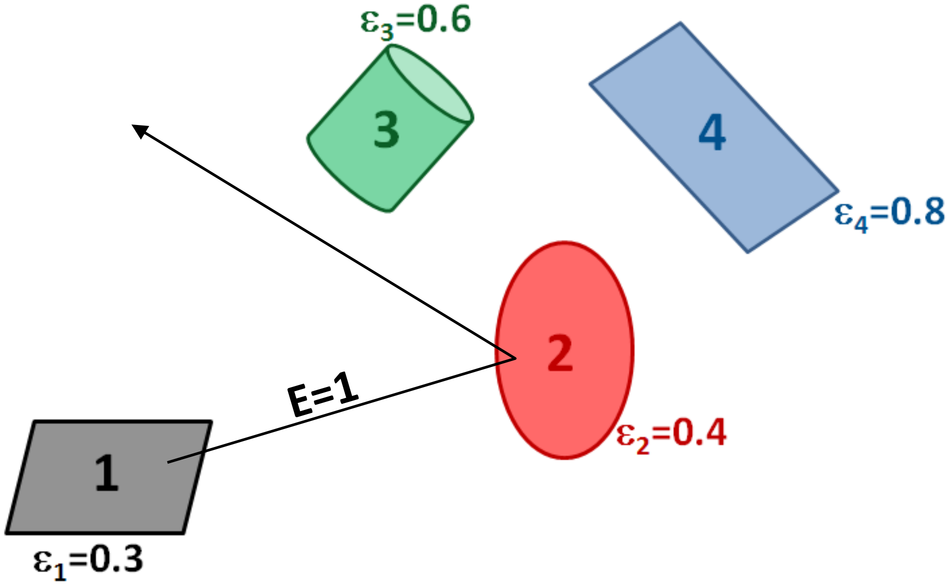
The rays hits Surface 2;

What is the energy transfer?



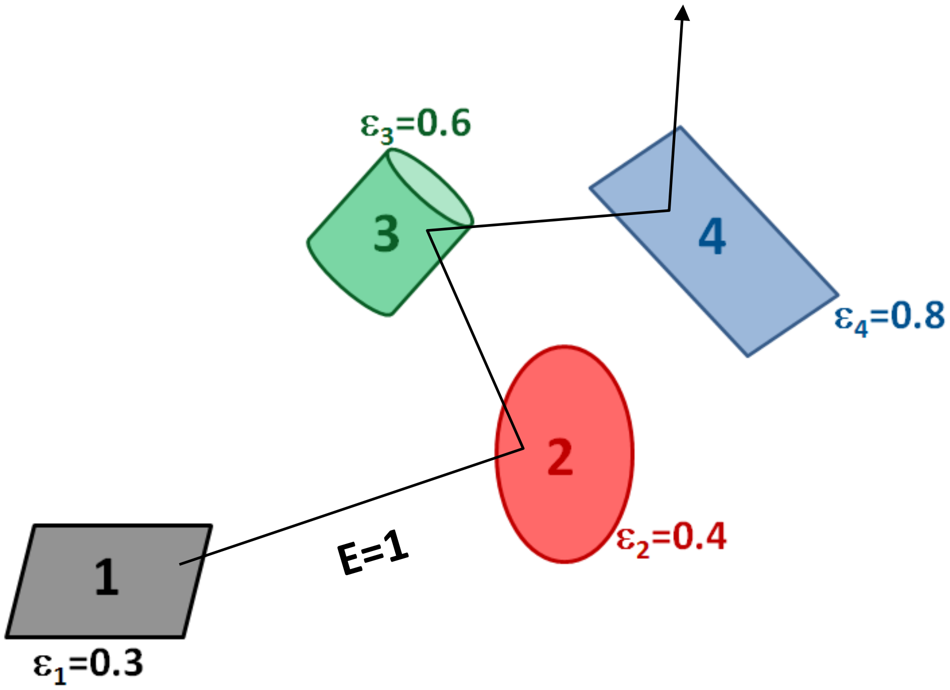
Calculating Grey Body Factors Using the Monte Carlo Method

Since the ray leaves Surface 1 with *unit energy* and hits a surface with an emittance of 0.4, we conclude that 40% of the energy is absorbed by Surface 2 and 60% is reflected.



Calculating Grey Body Factors Using the Monte Carlo Method

Now suppose another ray is shot from Surface 1 but, through reflections, strikes other surfaces in the model.

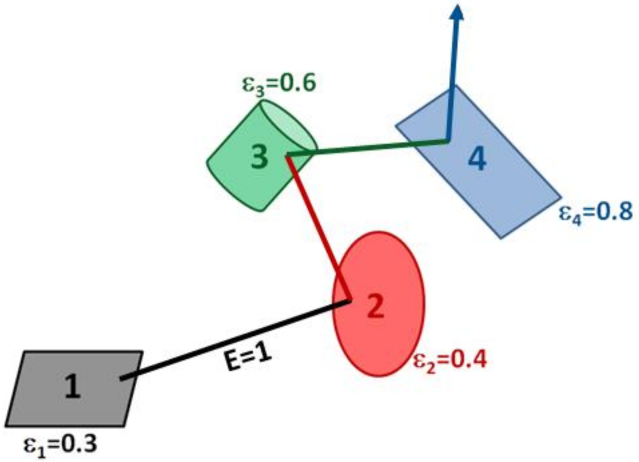


Calculating Grey Body Factors Using the Monte Carlo Method

We can tally the energy deposited as the ray moves from reflection to reflection:

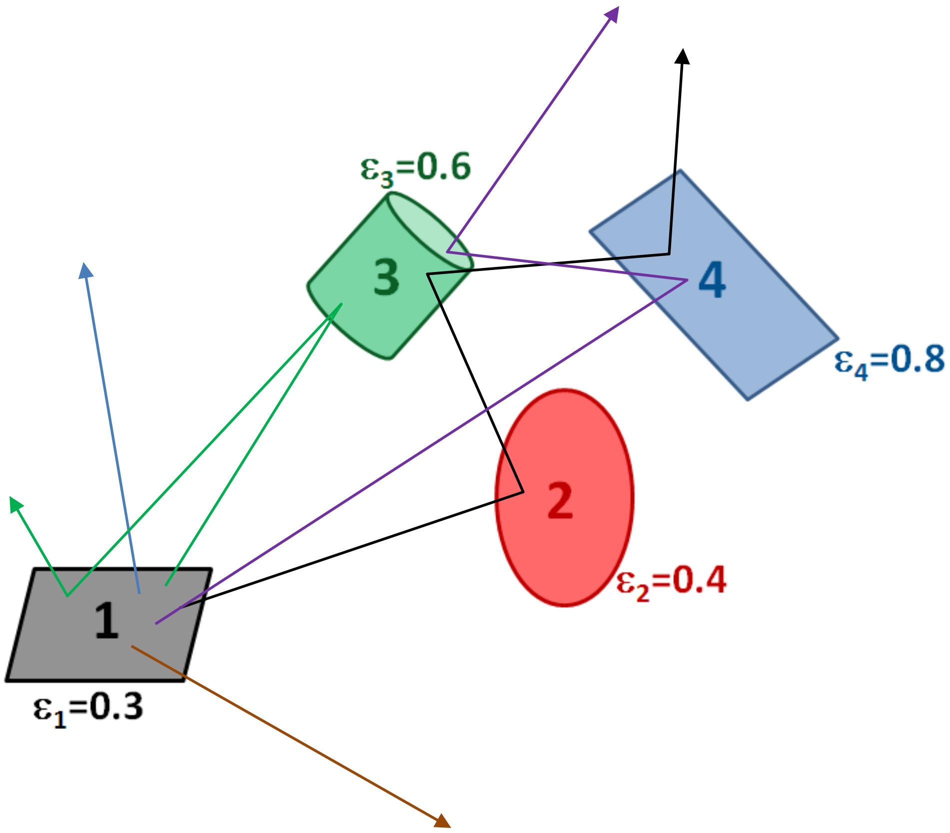
Path	Fraction of Original Energy *Deposited*	Fraction of Original Energy *Reflected*
1-2	0.400	0.600
2-3	0.360	0.240
3-4	0.192	0.048
4-Space	0.048	N/A

Total 1.000



Calculating Grey Body Factors Using the Monte Carlo Method

If many such rays are shot and a tally of energy deposition is maintained, we have an estimate of the fraction of energy leaving Surface 1 and arriving at Surface 2 via *all possible paths*.



Calculating Grey Body Factors Using the Monte Carlo Method

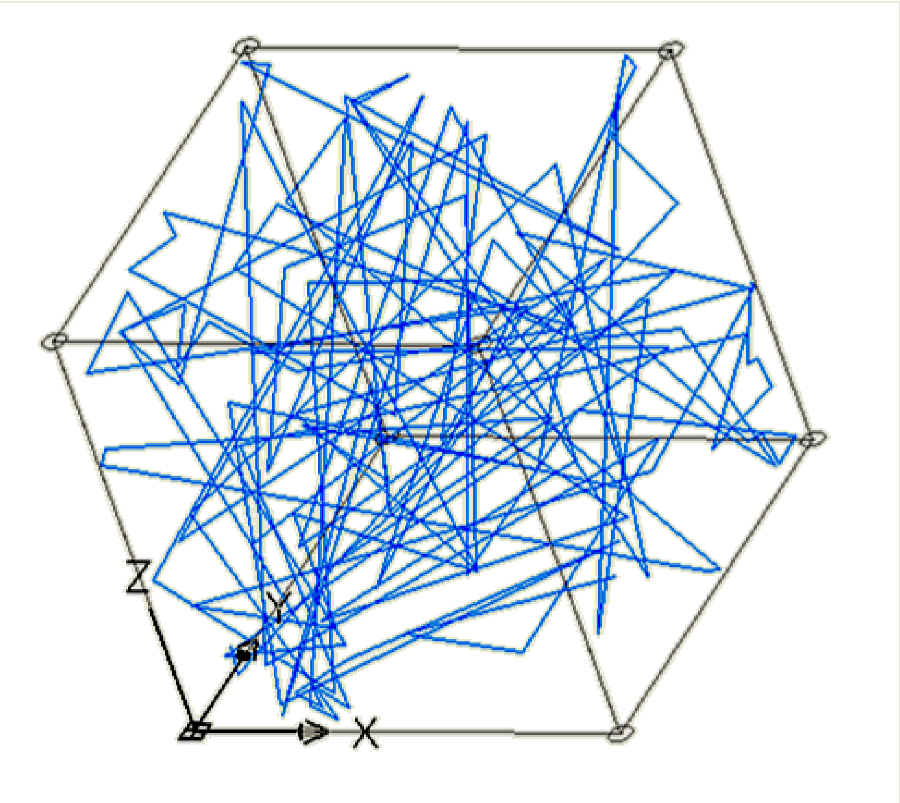
A ray is traced until it leaves the geometry or the ray energy drops below a specified threshold;

In the latter case, once this occurs, a test is performed at the next ray-to-surface intersection to determine whether the ray is completely absorbed or completely reflected -- depending on the surface properties;

This allows for energy conservation.

Calculating Grey Body Factors Using the Monte Carlo Method

In an enclosure, rays may undergo many bounces before the ray energy drops below the threshold to terminate the ray.

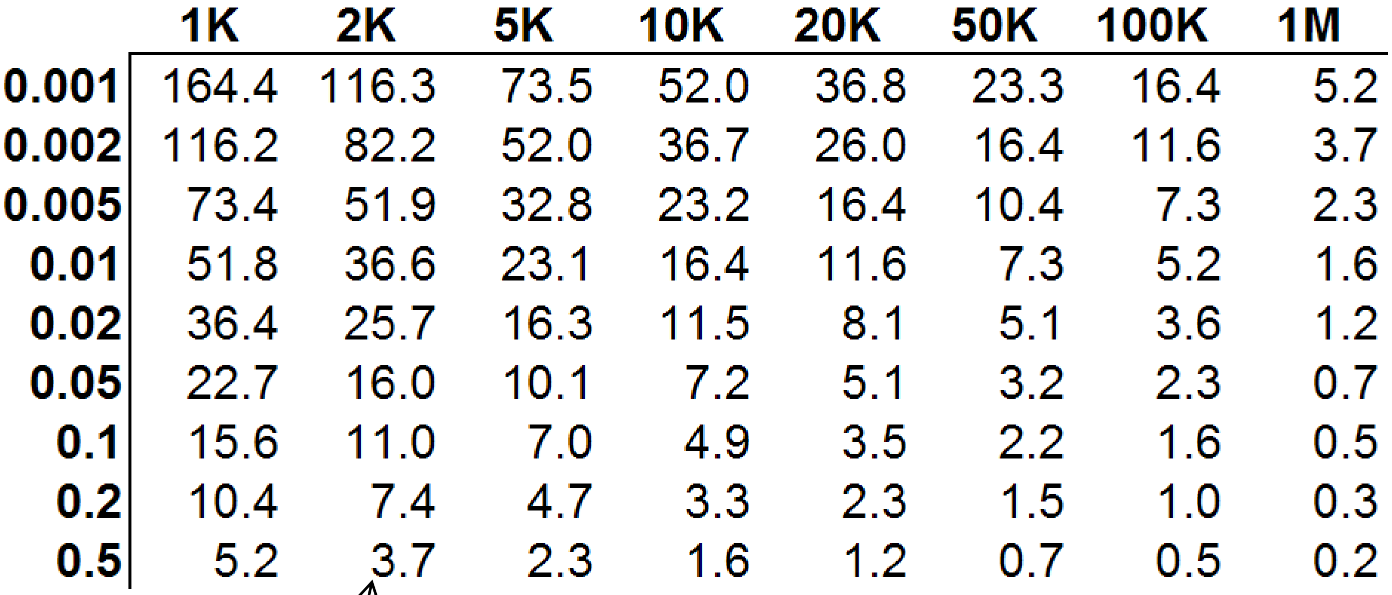


Single Ray Traced in Enclosure with ε = 0.05

Screen shot from RadCad® by Cullimore and Ring Technologies, Inc

Grey Body Statistical Error (Refs. 6 and 8)

A conservative error estimate is formed using discrete distribution assumptions also applies to grey body factors -- 90% confidence interval shown:



Number of Rays

Grey Body Factor Magnitude	1K	2K	5K	10K	20K	50K	100K	1M
0.001	164.4	116.3	73.5	52.0	36.8	23.3	16.4	5.2
0.002	116.2	82.2	52.0	36.7	26.0	16.4	11.6	3.7
0.005	73.4	51.9	32.8	23.2	16.4	10.4	7.3	2.3
0.01	51.8	36.6	23.1	16.4	11.6	7.3	5.2	1.6
0.02	36.4	25.7	16.3	11.5	8.1	5.1	3.6	1.2
0.05	22.7	16.0	10.1	7.2	5.1	3.2	2.3	0.7
0.1	15.6	11.0	7.0	4.9	3.5	2.2	1.6	0.5
0.2	10.4	7.4	4.7	3.3	2.3	1.5	1.0	0.3
0.5	5.2	3.7	2.3	1.6	1.2	0.7	0.5	0.2

% Error

The Fence Problem

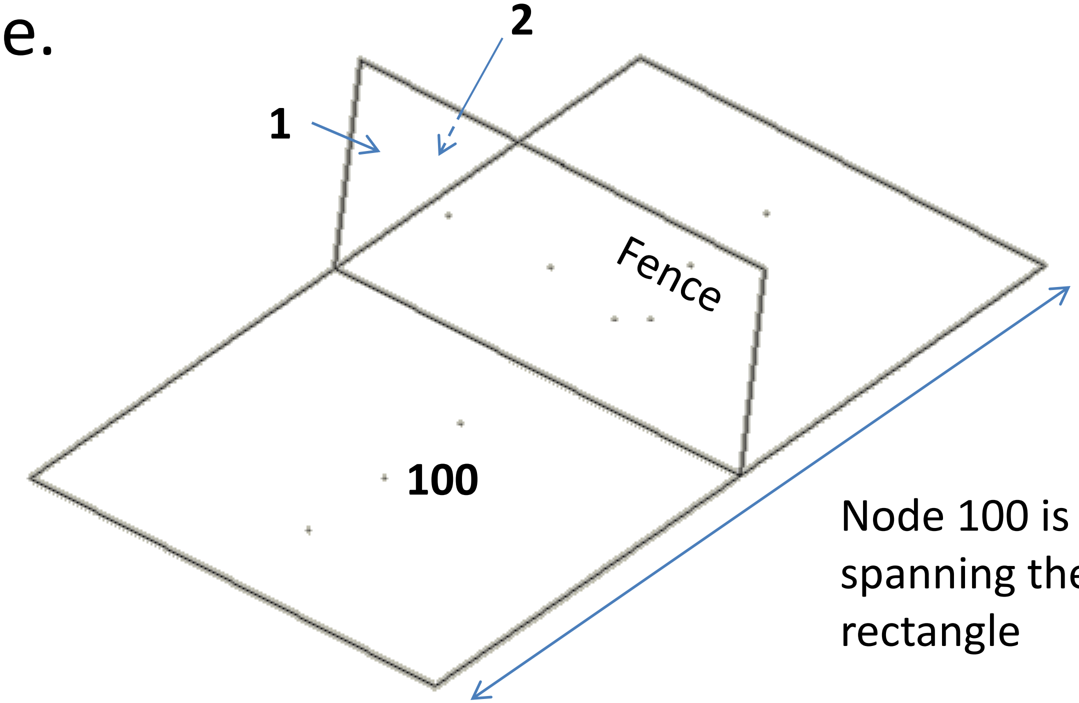
The Fence Problem

The *Fence Problem* arises when the Gebhart technique is used without giving proper consideration to nodal boundaries;

Let's examine how the Gebhart and Monte Carlo techniques handle this type of situation.

The Fence Problem

Consider the following geometry in which a fence is established at the midpoint of the long base rectangle.



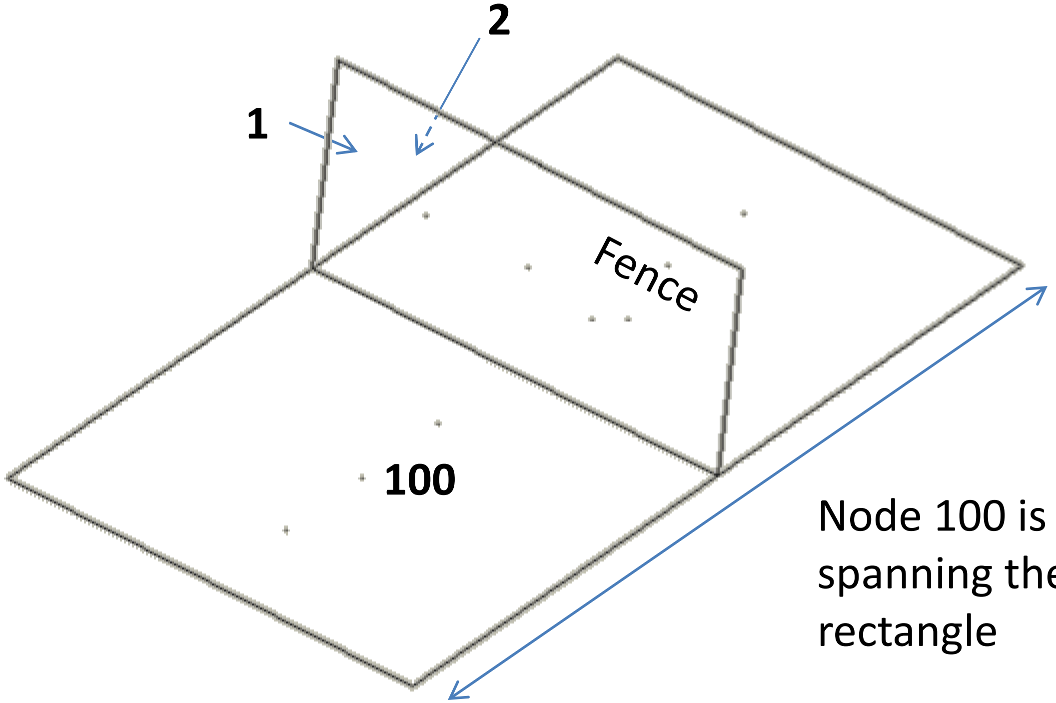
Node 100 is a single node spanning the entire base rectangle

All surface emittances, $\varepsilon = 0.5$.

Screen shot from RadCad® by Cullimore and Ring Technologies, Inc

The Fence Problem

Under normal circumstances, Nodes 1 and 2 should have no radiation coupling to one another.

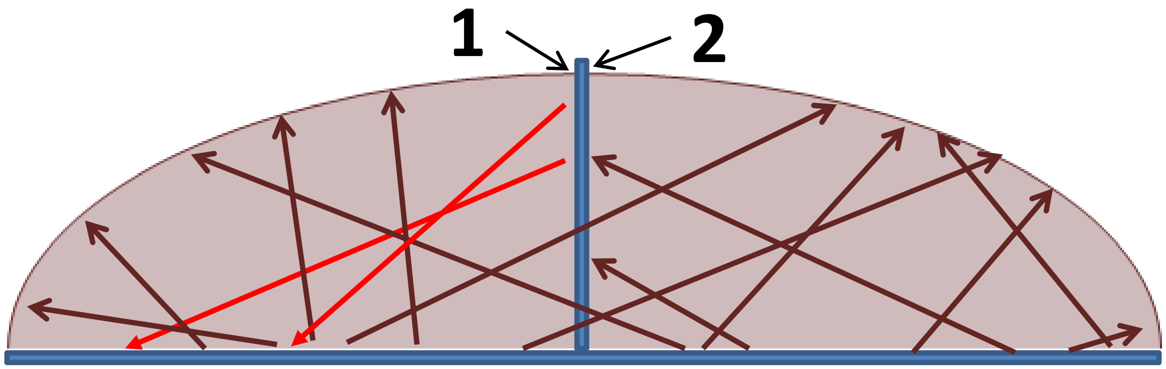


Node 100 is a single node spanning the entire base rectangle

Screen shot from RadCad® by Cullimore and Ring Technologies, Inc

The Fence Problem

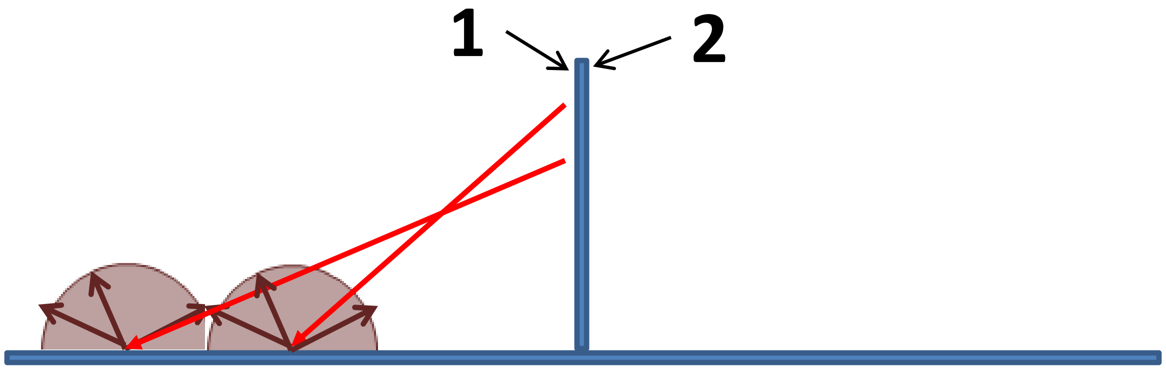
With Gebhart's method, energy incident onto the base rectangle is distributed and re-reflected from the *entire* area.



Single Base Rectangle Node

The Fence Problem

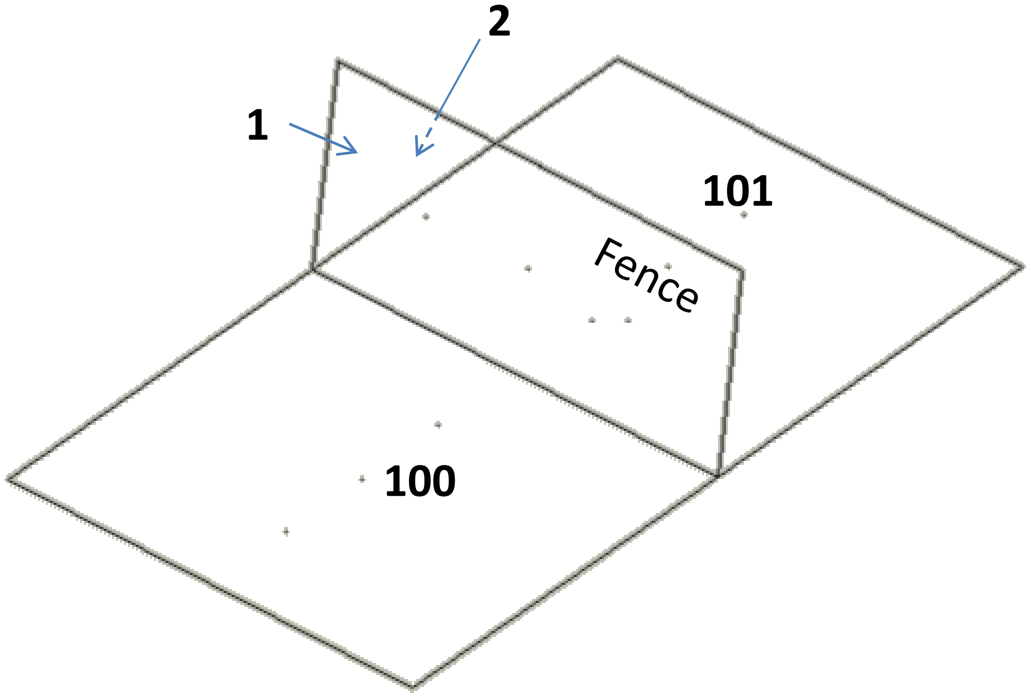
With a Monte Carlo (MC) approach, energy incident onto the base rectangle is re-reflected *locally*.



Single Base Rectangle Node (100)

The Fence Problem

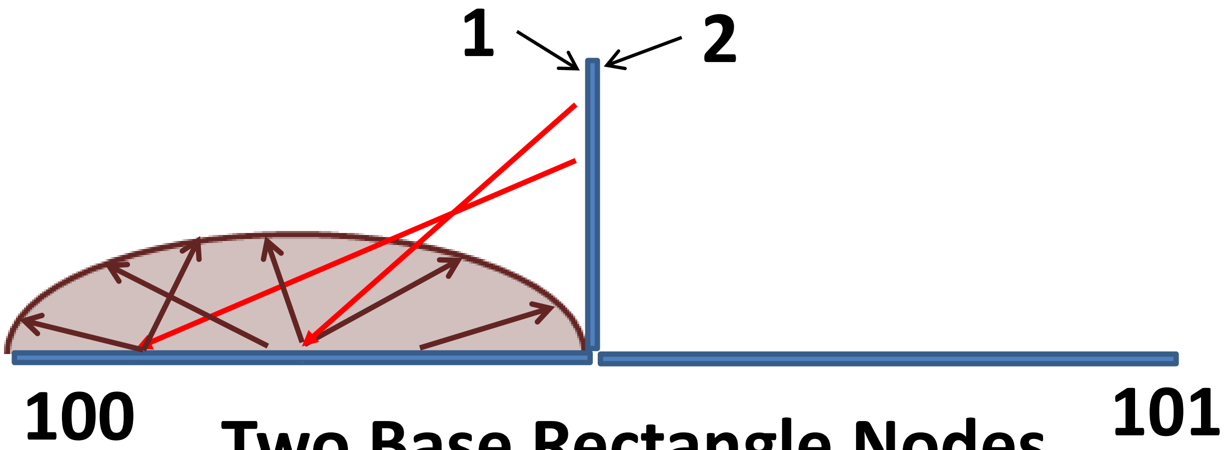
Now consider the same geometry but with the base rectangle divided into *two* nodes:



Screen shot from RadCad® by Cullimore and Ring Technologies, Inc

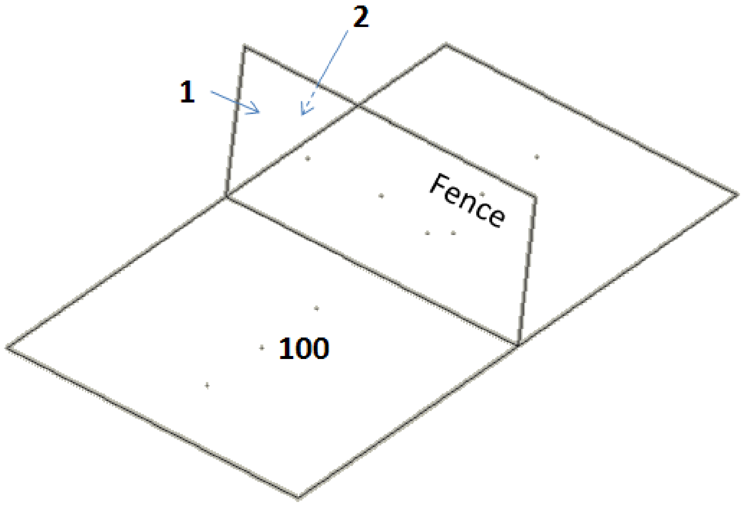
The Fence Problem

Using Gebhart's method with a properly nodalized geometry, leakage of energy to the other side of the fence is prevented.

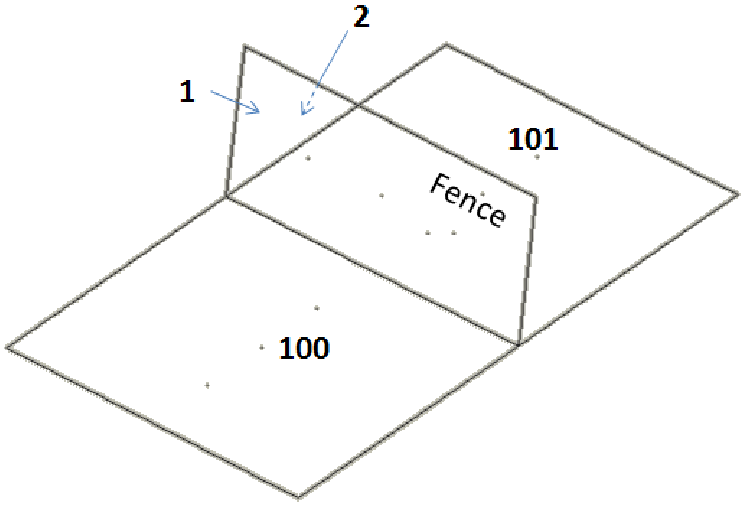


Two Base Rectangle Nodes

The Fence Problem



Single Base Rectangle



Base Rectangle Divided

Method	Base Rectangle Divided?	Radk$_{1\text{-}2}$
MC Form Factor + Gebhart	No	5.1744×10^{-4}
MC Form Factor + Gebhart	Yes	0
MC Grey Body Factors	No	0

Screen shot from RadCad® by Cullimore and Ring Technologies, Inc

The Fence Problem

Diffuse reflection using Gebhart's method across the entire base rectangle results in a grey body factor between Nodes 1 and 2;

When properly divided, the grey body factor between Nodes 1 and 2 is zero;

Monte Carlo-generated grey body factors rely on reflection from the ray-surface intersection point and does not show the same nodalization sensitivity as does the Gebhart approach.

Turning the Grey Body Factor into a Radk

Turning a Grey Body Factor into a Radk

We've seen how to generate grey body factors using, either, form factors plus Gebhart's method or by direct generation of grey bodies using Monte Carlo methods;

Now let's see how we can take the final step to turn the grey bodies into usable radiation conductances, or Radks.

Turning a Grey Body Factor into a Radk

The Radk $(G_{rad})_{ij}$ between two nodes *i* and *j* is given by:

$$(G_{rad})_{ij} = \varepsilon_i A_i B_{ij}$$

where ε_i is the emittance of node *i*, A_i is the area of node *i* and B_{ij} is the grey body factor from *i* to *j*.

We also note:

$$(G_{rad})_{ij} = (G_{rad})_{ji} = \varepsilon_j A_j B_{ji}$$

Radk Screening

For configurations with a large quantity of nodes and/or highly reflective surfaces with view factors to one another, it is possible to generate an undesirably large quantity of Radks;

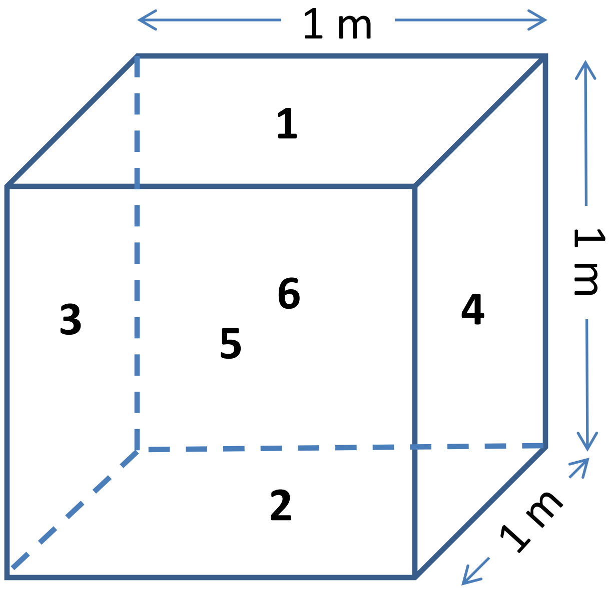
Engineers must judge which couplings are significant;

Radk dumps with different screening criteria may be used in thermal network models to determine the point of diminishing returns.

Application in Analysis

Example: Radiation in a Box

Consider the interior surfaces of a six-sided box with dimensions: 1m × 1m × 1m, and interior surface ε: 0.8



Side 1 is maintained at 100 C;
Sides 2 - 6 are maintained at 20 C;

Considering only radiation,
what is the heat transfer between Side 1 and the other box sides?

Example: Radiation in a Box

The form factor solution for opposing parallel plates (Sides 1 and 2) yields an exact answer of 0.2;

For the form factors to sum to unity, the form factor sum from Side 1 to the remaining Sides (3, 4, 5, 6) must be 0.8 (i.e., 1 - 0.2). By symmetry arguments:

$$FF_{13} = FF_{14} = FF_{15} = FF_{16} = 0.2$$

Example: Radiation in a Box

If we wish to solve this using Gebhart's method, the system of equations for Side 1 will be:

$$B_{11} = \varepsilon_1 FF_{11} + (1-\varepsilon_1)FF_{11}B_{11} + (1-\varepsilon_2)FF_{12}B_{21} + \cdots + (1-\varepsilon_6)FF_{16}B_{61}$$

$$B_{12} = \varepsilon_1 FF_{12} + (1-\varepsilon_1)FF_{11}B_{12} + (1-\varepsilon_2)FF_{12}B_{22} + \cdots + (1-\varepsilon_6)FF_{16}B_{62}$$

$$B_{13} = \varepsilon_1 FF_{13} + (1-\varepsilon_1)FF_{11}B_{13} + (1-\varepsilon_2)FF_{12}B_{23} + \cdots + (1-\varepsilon_6)FF_{16}B_{63}$$

$$B_{14} = \varepsilon_1 FF_{14} + (1-\varepsilon_1)FF_{11}B_{14} + (1-\varepsilon_2)FF_{12}B_{24} + \cdots + (1-\varepsilon_6)FF_{16}B_{64}$$

$$B_{15} = \varepsilon_1 FF_{15} + (1-\varepsilon_1)FF_{11}B_{15} + (1-\varepsilon_2)FF_{12}B_{25} + \cdots + (1-\varepsilon_6)FF_{16}B_{65}$$

$$B_{16} = \varepsilon_1 FF_{16} + (1-\varepsilon_1)FF_{11}B_{16} + (1-\varepsilon_2)FF_{13}B_{26} + \cdots + (1-\varepsilon_6)FF_{16}B_{66}$$

Example: Radiation in a Box

Rearrange the equations into the form:

$$\left[\quad \varepsilon, FF \quad\right]\begin{Bmatrix} B_{11} \\ \vdots \\ B_{16} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_1 FF_{11} \\ \vdots \\ \varepsilon_1 FF_{16} \end{Bmatrix}$$

Substituting known values, we have this system:

$$\begin{bmatrix} 1 & -0.04 & -0.04 & -0.04 & -0.04 & -0.04 \\ -0.04 & 1 & -0.04 & -0.04 & -0.04 & -0.04 \\ -0.04 & -0.04 & 1 & -0.04 & -0.04 & -0.04 \\ -0.04 & -0.04 & -0.04 & 1 & -0.04 & -0.04 \\ -0.04 & -0.04 & -0.04 & -0.04 & 1 & -0.04 \\ -0.04 & -0.04 & -0.04 & -0.04 & -0.04 & 1 \end{bmatrix} \begin{Bmatrix} B_{11} \\ B_{12} \\ B_{13} \\ B_{14} \\ B_{15} \\ B_{16} \end{Bmatrix} = \begin{Bmatrix} 0.00 \\ 0.16 \\ 0.16 \\ 0.16 \\ 0.16 \\ 0.16 \end{Bmatrix}$$

Example: Radiation in a Box

Our B_{ij} values are:

$B_{11} = 0.038462$
$B_{12} = 0.192308$
$B_{13} = 0.192308$
$B_{14} = 0.192308$
$B_{15} = 0.192308$
$B_{16} = 0.192308$

To obtain our Radks, we multiply by the area and the emittance of Side 1.

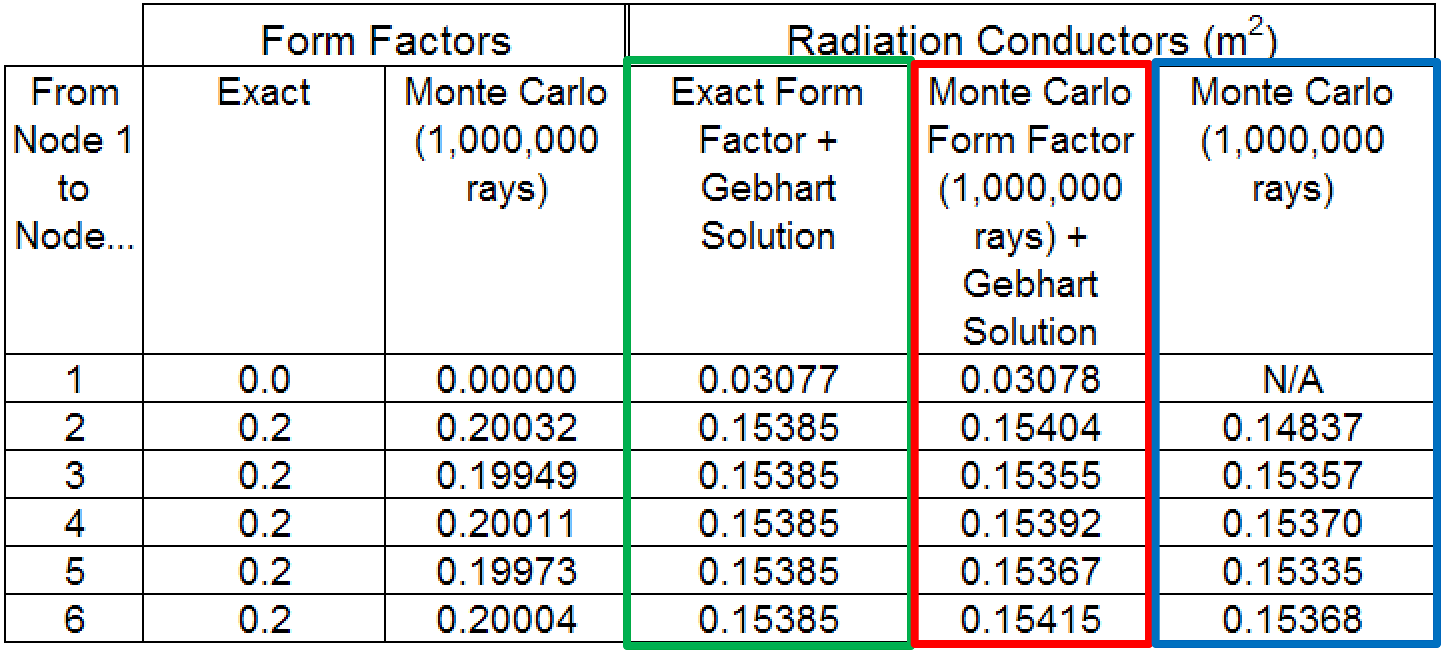
Example: Radiation in a Box

We can also use Gebhart's method using Monte Carlo-generated form factors -- we'll present this solution, as well, using 1,000,000 rays per node (and we'll assume reciprocity to find F_{ji} given F_{ij});

Finally, we'll form our Radks directly using Monte Carlo-generated grey bodies, again using 1,000,000 rays per node.

Example: Radiation in a Box

Comparing solutions...



	Form Factors		Radiation Conductors (m^2)		
From Node 1 to Node...	Exact	Monte Carlo (1,000,000 rays)	Exact Form Factor + Gebhart Solution	Monte Carlo Form Factor (1,000,000 rays) + Gebhart Solution	Monte Carlo (1,000,000 rays)
1	0.0	0.00000	0.03077	0.03078	N/A
2	0.2	0.20032	0.15385	0.15404	0.14837
3	0.2	0.19949	0.15385	0.15355	0.15357
4	0.2	0.20011	0.15385	0.15392	0.15370
5	0.2	0.19973	0.15385	0.15367	0.15335
6	0.2	0.20004	0.15385	0.15415	0.15368

Example: Radiation in a Box

The heat flow is...

$$\dot{Q}_{12} = \sigma \varepsilon_1 A_1 B_{12} (T_1^4 - T_2^4)$$

$$\dot{Q}_{12} = \sigma G_{12} (T_1^4 - T_2^4)$$

$$\dot{Q}_{12} = \left(5.67 \times 10^{-8} \frac{W}{m^2 K^4}\right)(0.15385\ m^2)[(373\ K)^4 - (293\ K)^4]$$

$$\boxed{\dot{Q}_{12} = 104.6\ W}$$

By similarity arguments, the heat transfer between Side 1 and all other sides is the same.

Concluding Remarks

A discussion of form factors, grey body factors and radiation conductors has been presented;

Analytical formulations were shown for a variety of methodologies;

Sample problems were provided to aid in understanding.

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Monte Carlo ray trace analyses and screen shots from RadCad® by Cullimore and Ring Technologies, Inc.

Microsoft® clip art was used throughout this lesson.

For Additional Information

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