AN INNOVATIVE METHODOLOGY FOR ERROR ANALYSIS OF THERMO-FLUID SYSTEMS

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Governing Equations

- *Conservation of Mass :*

$$\frac{\partial(\rho A)}{\partial t} + \frac{\partial(\rho u A)}{\partial x} = 0$$

- Conservation of Momentum :

$$\frac{\partial(\rho u A)}{\partial t} + \frac{\partial[(\rho u^2 + p)A]}{\partial x} - p\frac{dA}{dx} = 0$$

- Conservation of Energy :

$$\frac{\partial(\rho e_T A)}{\partial t} + \frac{\partial[(\rho e_T + p)uA]}{\partial x} = 0$$

Governing Equations- Non Dimensional

- *Conservation of Mass :*

$$\frac{\partial(\bar{\rho}\bar{A})}{\partial\bar{t}} + \frac{\partial(\bar{\rho}\bar{u}\bar{A})}{\partial\bar{x}} = 0$$

- Conservation of Momentum :

$$\frac{\partial(\bar{\rho}\bar{u}\bar{A})}{\partial\bar{t}} + \frac{\partial}{\partial\bar{x}}\left[\bar{\rho}\bar{A}\left(\bar{u}^2 + \frac{\bar{T}}{\gamma}\right)\right] - \frac{\bar{\rho}\bar{T}}{\gamma}\frac{d\bar{A}}{d\bar{x}} = 0$$

- Conservation of Energy :

$$\frac{\partial}{\partial\bar{t}}(\bar{\rho}\bar{e}_T\bar{A}) + \frac{\partial}{\partial\bar{x}}[\bar{\rho}\bar{u}\bar{A}(\bar{e}_T + \bar{T})] = 0$$

Where $\bar{e}_T \stackrel{\text{def}}{=} \frac{\bar{e}}{(\gamma-1)} + \frac{\gamma}{2}\bar{u}^2$

Vector Form

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} - G = 0$$

Where the vectors, U, F and G are defined as

$$U = \begin{bmatrix} \rho A \\ \rho u A \\ \rho e_T A \end{bmatrix} \quad , \quad F = \begin{bmatrix} \rho u A \\ \rho A \left(u^2 + \frac{T}{\gamma} \right) \\ \rho u A (e_T + T) \end{bmatrix}, \; G = \begin{bmatrix} 0 \\ \frac{\rho T}{\gamma} \frac{dA}{dx} \\ 0 \end{bmatrix}$$

Problem ID1: Anderson Nozzle Isentropic Flow

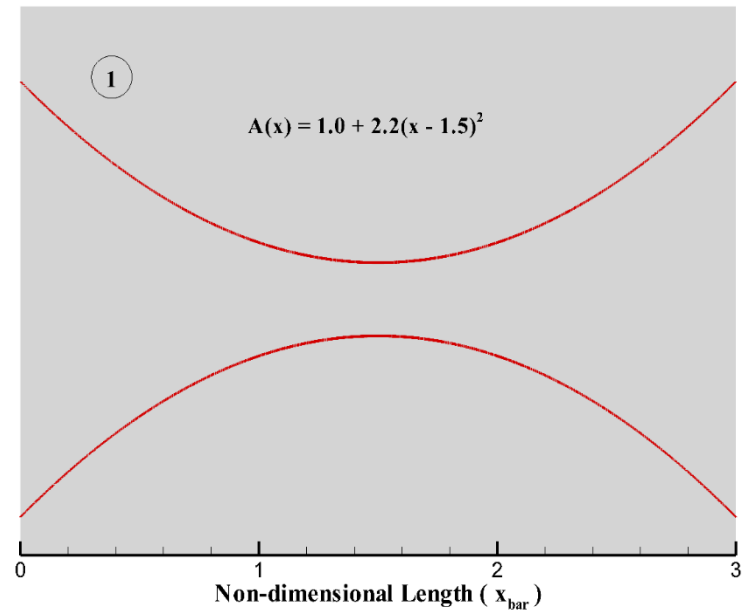
Initial Conditions

$$\begin{bmatrix}\rho\\ u\\ T\end{bmatrix}_{x_{bar}}^{t=0} = \begin{bmatrix}1.0\\ 0.59/\rho A\\ 1.0\end{bmatrix}_{x_{bar}} \qquad 0 \le x_{bar} < 0.5$$

$$\begin{bmatrix}\rho\\ u\\ T\end{bmatrix}_{x_{bar}}^{t=0} = \begin{bmatrix}1.0 - 0.366(x_{bar} - 0.5)\\ 0.59/\rho A\\ 1.0 - 0.167(x_{bar} - 0.5)\end{bmatrix}_{x_{bar}} \qquad 0.5 \le x_{bar} < 1.5$$

$$\begin{bmatrix}\rho\\ u\\ T\end{bmatrix}_{x_{bar}}^{t=0} = \begin{bmatrix}0.634 - 0.3879(x_{bar} - 1.5)\\ 0.59/\rho A\\ 0.833 - 0.3507(x_{bar} - 1.5)\end{bmatrix}_{x_{bar}} \qquad x_{bar} \ge 1.5$$

$$A(x) = 1.0 + 2.2(x - 1.5)^2$$



Problem ID1: Anderson Nozzle Isentropic Flow

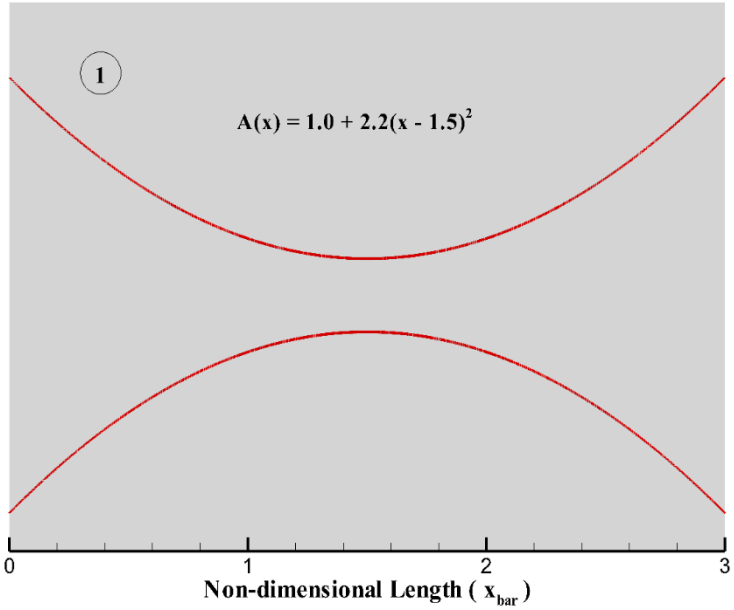
Inflow Conditions

$$\begin{bmatrix}\rho\\ u\\ T\end{bmatrix}_{i=1}^{t} = \begin{bmatrix}1.0\\ 2.0u_2 - u_3\\ 1.0\end{bmatrix}$$

Outflow Conditions

$$\begin{bmatrix}\rho\\ u\\ T\end{bmatrix}_{i_{max}}^{n} = \begin{bmatrix}2.0\rho_{(i_{max}-1)} - \rho_{(i_{max}-2)}\\ 2.0u_{(i_{max}-1)} - u_{(i_{max}-2)}\\ 2.0T_{(i_{max}-1)} - T_{(i_{max}-2)}\end{bmatrix}$$

$$A(x) = 1.0 + 2.2(x - 1.5)^2$$



Problem ID2: Anderson Nozzle Shock Flow

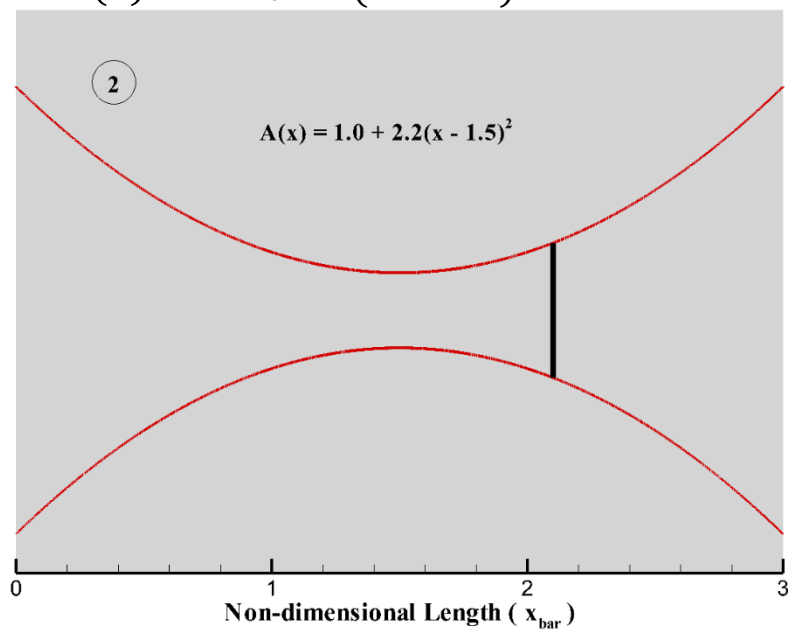
Initial Conditions

$$\begin{bmatrix}\rho\\ u\\ T\end{bmatrix}_{x_{bar}}^{t=0} = \begin{bmatrix}1.0\\ 0.59/\rho A\\ 1.0\end{bmatrix}_{x_{bar}} \qquad 0 \le x_{bar} < 0.5$$

$$\begin{bmatrix}\rho\\ u\\ T\end{bmatrix}_{x_{bar}}^{t=0} = \begin{bmatrix}1.0 - 0.366(x_{bar} - 0.5)\\ 0.59/\rho A\\ 1.0 - 0.167(x_{bar} - 0.5)\end{bmatrix}_{x_{bar}} \qquad 0.5 \le x_{bar} < 1.5$$

$$\begin{bmatrix}\rho\\ u\\ T\end{bmatrix}_{x_{bar}}^{t=0} = \begin{bmatrix}0.634 - 0.3879(x_{bar} - 1.5)\\ 0.59/\rho A\\ 0.833 - 0.3507(x_{bar} - 1.5)\end{bmatrix}_{x_{bar}} \qquad x_{bar} \ge 1.5$$

$$A(x) = 1.0 + 2.2(x - 1.5)^2$$



Problem ID2: Anderson Nozzle Shock Flow

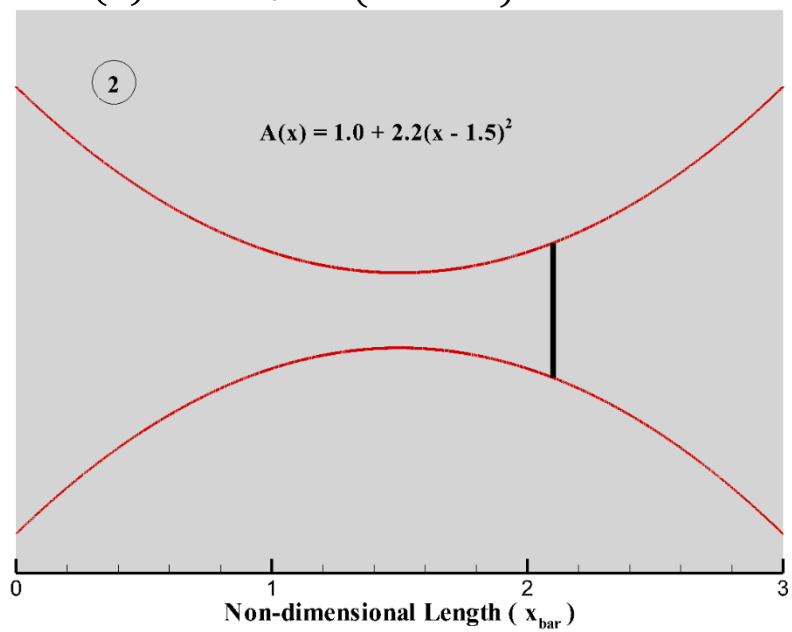
Inflow Conditions

$$\begin{bmatrix} \rho \\ u \\ T \end{bmatrix}_{i=1}^{t} = \begin{bmatrix} 1.0 \\ 2.0u_2 - u_3 \\ 1.0 \end{bmatrix}$$

Outflow Conditions

$$\begin{bmatrix} \rho \\ u \\ T \end{bmatrix}_{i_{max}}^{t} = \begin{bmatrix} \rho_{i_{max}-1} \\ 0.1520 \\ T_{i_{max}-1} \end{bmatrix}$$

$$A(x) = 1.0 + 2.2(x - 1.5)^2$$



Problem ID3: Absolute Nozzle Isentropic Flow

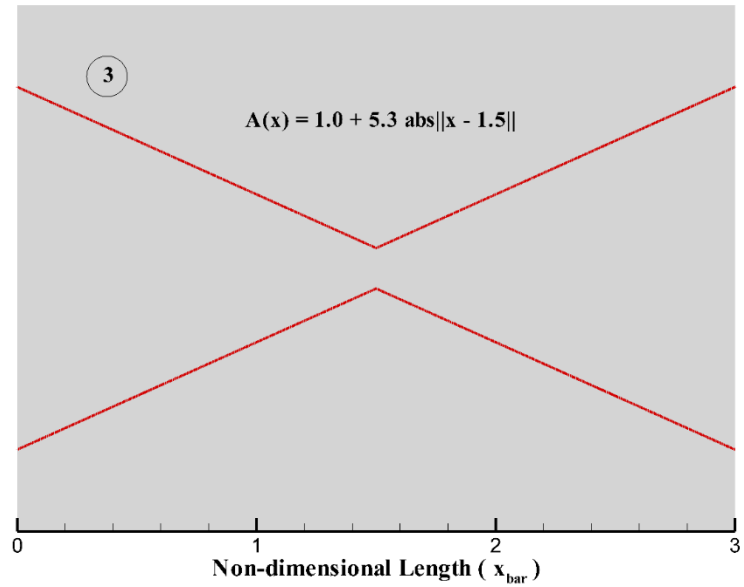
Initial Conditions

$$\begin{bmatrix}\rho\\ u\\ T\end{bmatrix}_{x_{bar}}^{t=0} = \begin{bmatrix}1.0\\ 0.59/\rho A\\ 1.0\end{bmatrix}_{x_{bar}} \qquad 0 \le x_{bar} < 0.5$$

$$\begin{bmatrix}\rho\\ u\\ T\end{bmatrix}_{x_{bar}}^{t=0} = \begin{bmatrix}1.0 - 0.366(x_{bar} - 0.5)\\ 0.59/\rho A\\ 1.0 - 0.167(x_{bar} - 0.5)\end{bmatrix}_{x_{bar}} \qquad 0.5 \le x_{bar} < 1.5$$

$$\begin{bmatrix}\rho\\ u\\ T\end{bmatrix}_{x_{bar}}^{t=0} = \begin{bmatrix}0.634 - 0.3879(x_{bar} - 1.5)\\ 0.59/\rho A\\ 0.833 - 0.3507(x_{bar} - 1.5)\end{bmatrix}_{x_{bar}} \qquad x_{bar} \ge 1.5$$

$$A(x) = 1.0 + 5.3\, abs||x - 1.5||$$



Problem ID3: Absolute Nozzle Isentropic Flow

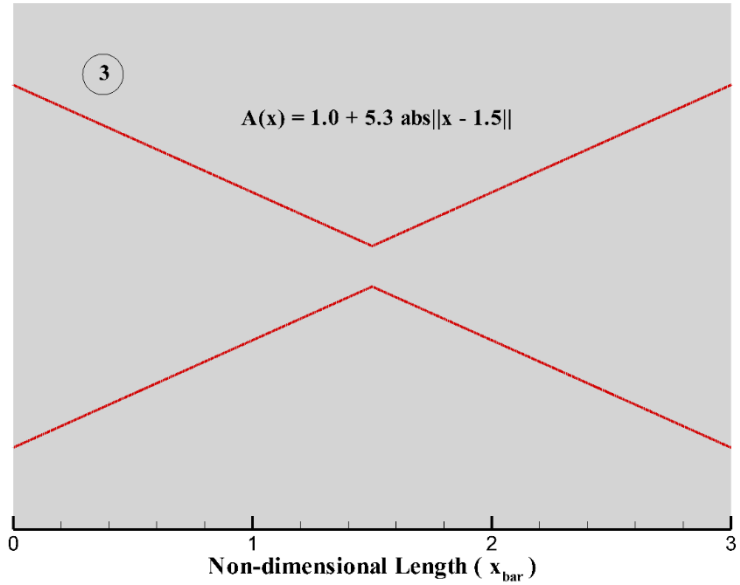
Inflow Conditions

$$\begin{bmatrix}\rho\\ u\\ T\end{bmatrix}_{i=1}^{t} = \begin{bmatrix}1.0\\ 2.0u_2 - u_3\\ 1.0\end{bmatrix}$$

Outflow Conditions

$$\begin{bmatrix}\rho\\ u\\ T\end{bmatrix}_{i_{max}}^{n} = \begin{bmatrix}2.0\rho_{(i_{max}-1)} - \rho_{(i_{max}-2)}\\ 2.0u_{(i_{max}-1)} - u_{(i_{max}-2)}\\ 2.0T_{(i_{max}-1)} - T_{(i_{max}-2)}\end{bmatrix}$$

$$A(x) = 1.0 + 5.3\ abs||x\ - 1.5||$$



Problem ID4: Absolute Nozzle Isentropic Flow

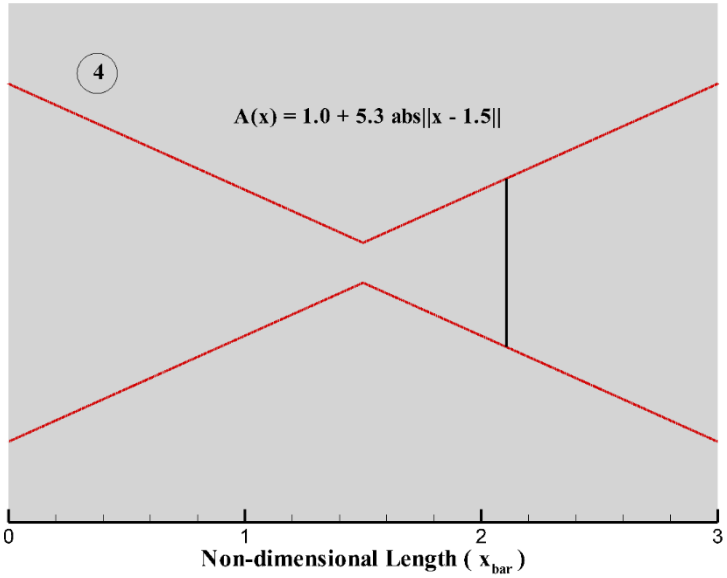
Initial Conditions

$$\begin{bmatrix}\rho\\ u\\ T\end{bmatrix}_{x_{bar}}^{t=0} = \begin{bmatrix}1.0\\ 0.59/\rho A\\ 1.0\end{bmatrix}_{x_{bar}} \qquad 0 \le x_{bar} < 0.5$$

$$\begin{bmatrix}\rho\\ u\\ T\end{bmatrix}_{x_{bar}}^{t=0} = \begin{bmatrix}1.0 - 0.366(x_{bar} - 0.5)\\ 0.59/\rho A\\ 1.0 - 0.167(x_{bar} - 0.5)\end{bmatrix}_{x_{bar}} \qquad 0.5 \le x_{bar} < 1.5$$

$$\begin{bmatrix}\rho\\ u\\ T\end{bmatrix}_{x_{bar}}^{t=0} = \begin{bmatrix}0.634 - 0.3879(x_{bar} - 1.5)\\ 0.59/\rho A\\ 0.833 - 0.3507(x_{bar} - 1.5)\end{bmatrix}_{x_{bar}} \qquad x_{bar} \ge 1.5$$

$$A(x) = 1.0 + 5.3\, abs||x - 1.5||$$



Problem ID4: Absolute Nozzle Isentropic Flow

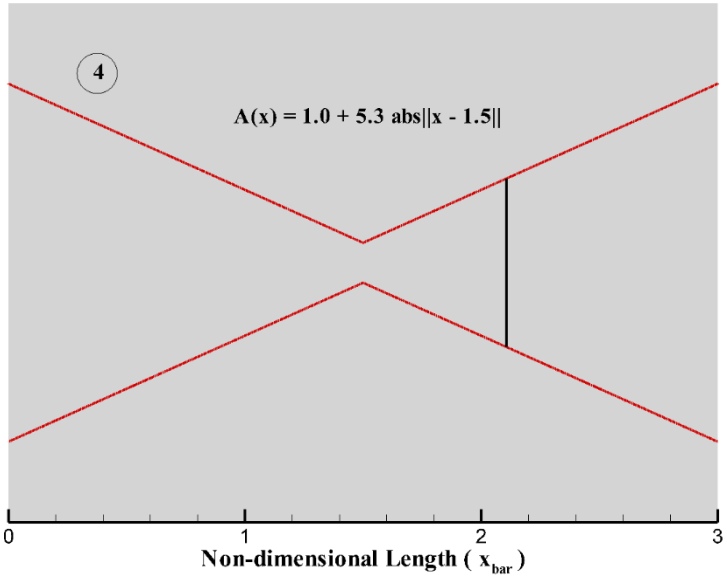
Inflow Conditions

$$\begin{bmatrix} \rho \\ u \\ T \end{bmatrix}_{i=1}^{t} = \begin{bmatrix} 1.0 \\ 2.0u_2 - u_3 \\ 1.0 \end{bmatrix}$$

Outflow Conditions

$$\begin{bmatrix} \rho \\ u \\ T \end{bmatrix}_{i_{max}}^{t} = \begin{bmatrix} \rho_{i_{max}-1} \\ 0.1520 \\ T_{i_{max}-1} \end{bmatrix}$$

$$A(x) = 1.0 + 5.3\ abs||x\ - 1.5||$$



Problem ID5: Feng Nozzle Isentropic Flow

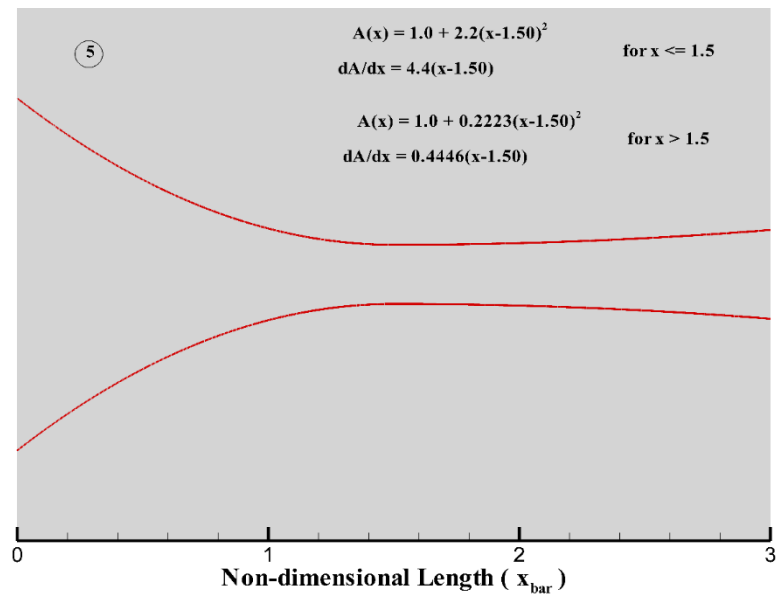
Initial Conditions

$$A(x) = 1.0 + \ 2.2(x - 1.50)^2 \qquad for\ x \le 1.5$$
$$A(x) = 1.0 + 0.2223(x - 1.50)^2 \qquad for\ x > 1.5$$

$$\begin{bmatrix}\rho\\ u\\ T\end{bmatrix}_{x_{bar}}^{t=0} = \begin{bmatrix}1.0\\ 0.59/\rho A\\ 1.0\end{bmatrix}_{x_{bar}} \qquad 0 \le x_{bar} < 0.5$$

$$\begin{bmatrix}\rho\\ u\\ T\end{bmatrix}_{x_{bar}}^{t=0} = \begin{bmatrix}1.0 - 0.366(x_{bar} - 0.5)\\ 0.59/\rho A\\ 1.0 - 0.167(x_{bar} - 0.5)\end{bmatrix}_{x_{bar}} \qquad 0.5 \le x_{bar} < 1.5$$

$$\begin{bmatrix}\rho\\ u\\ T\end{bmatrix}_{x_{bar}}^{t=0} = \begin{bmatrix}0.634 - 0.3879(x_{bar} - 1.5)\\ 0.59/\rho A\\ 0.833 - 0.3507(x_{bar} - 1.5)\end{bmatrix}_{x_{bar}} \qquad x_{bar} \ge 1.5$$



Problem ID5: Feng Nozzle Isentropic Flow

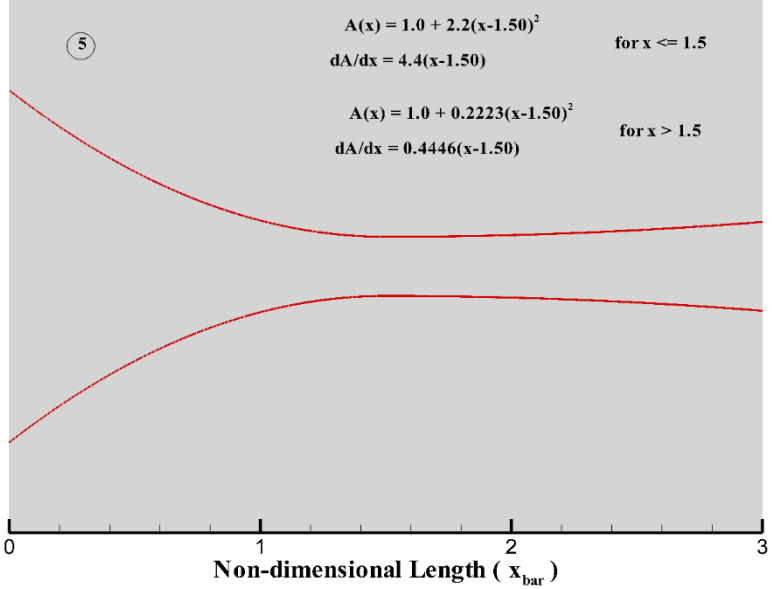
Inflow Conditions

$$\begin{bmatrix}\rho\\ u\\ T\end{bmatrix}_{i=1}^{t} = \begin{bmatrix}1.0\\ 2.0u_2 - u_3\\ 1.0\end{bmatrix}$$

Outflow Conditions

$$\begin{bmatrix}\rho\\ u\\ T\end{bmatrix}_{i_{max}}^{n} = \begin{bmatrix}2.0\rho_{(i_{max}-1)} - \rho_{(i_{max}-2)}\\ 2.0u_{(i_{max}-1)} - u_{(i_{max}-2)}\\ 2.0T_{(i_{max}-1)} - T_{(i_{max}-2)}\end{bmatrix}$$

$$A(x) = 1.0 + \ 2.2(x - 1.50)^2 \qquad for\ x \leq 1.5$$
$$A(x) = 1.0 + 0.2223(x - 1.50)^2 \qquad for\ x > 1.5$$



Problem ID6: Feng Nozzle Shock Flow

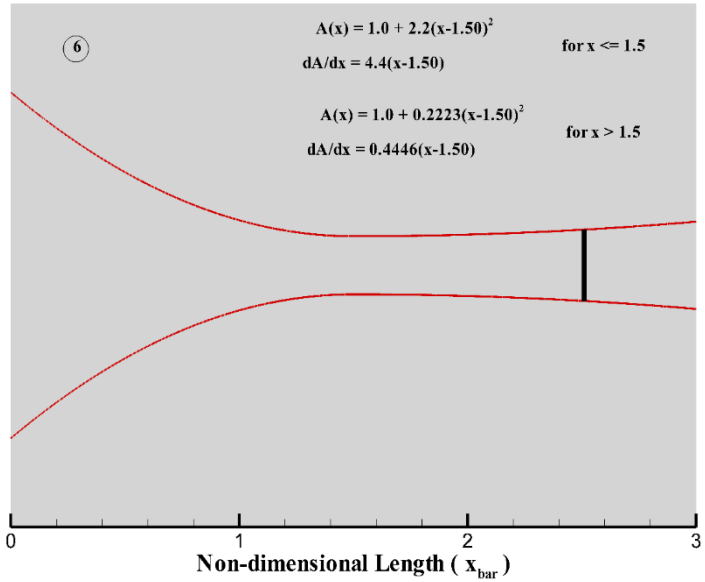
Initial Conditions

$$A(x) = 1.0 + \ 2.2(x - 1.50)^2 \qquad for\ x \le 1.5$$
$$A(x) = 1.0 + 0.2223(x - 1.50)^2 \qquad for\ x > 1.5$$

$$\begin{bmatrix} \rho \\ u \\ T \end{bmatrix}_{x_{bar}}^{t=0} = \begin{bmatrix} 1.0 \\ 0.59/\rho A \\ 1.0 \end{bmatrix}_{x_{bar}} \qquad 0 \le x_{bar} < 0.5$$

$$\begin{bmatrix} \rho \\ u \\ T \end{bmatrix}_{x_{bar}}^{t=0} = \begin{bmatrix} 1.0 - 0.366(x_{bar} - 0.5) \\ 0.59/\rho A \\ 1.0 - 0.167(x_{bar} - 0.5) \end{bmatrix}_{x_{bar}} \qquad 0.5 \le x_{bar} < 1.5$$

$$\begin{bmatrix} \rho \\ u \\ T \end{bmatrix}_{x_{bar}}^{t=0} = \begin{bmatrix} 0.634 - 0.3879(x_{bar} - 1.5) \\ 0.59/\rho A \\ 0.833 - 0.3507(x_{bar} - 1.5) \end{bmatrix}_{x_{bar}} \qquad x_{bar} \ge 1.5$$



Problem ID6: Feng Nozzle Shock Flow

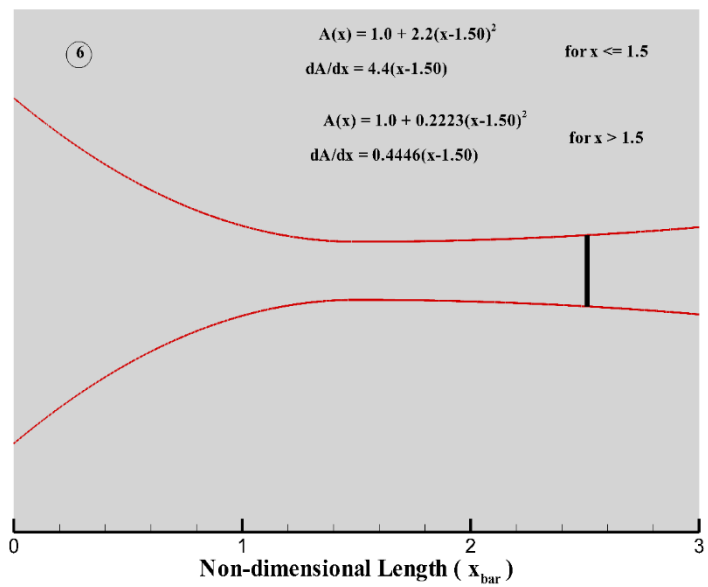
Inflow Conditions

$$\begin{bmatrix}\rho\\ u\\ T\end{bmatrix}_{i=1}^{t} = \begin{bmatrix}1.0\\ 2.0u_2 - u_3\\ 1.0\end{bmatrix}$$

Outflow Conditions

$$\begin{bmatrix}\rho\\ u\\ T\end{bmatrix}_{i_{max}}^{n} = \begin{bmatrix}\rho_{(i_{max}-1)}\\ 0.48\\ T_{(i_{max}-1)}\end{bmatrix}$$

$$A(x) = 1.0 + \ 2.2(x - 1.50)^2 \qquad for\ x \le 1.5$$
$$A(x) = 1.0 + 0.2223(x - 1.50)^2 \qquad for\ x > 1.5$$

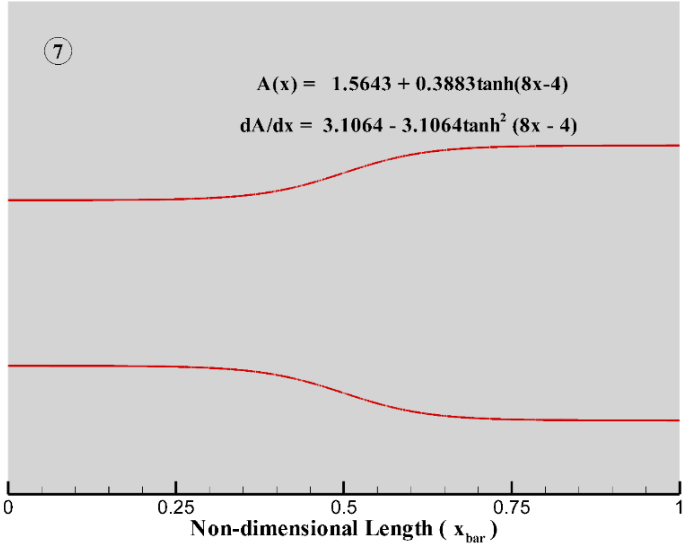


Problem ID7: Hoffmann Nozzle Isentropic flow

Initial Conditions

$$A(x) = 1.5643 + 0.3883\tanh(8x - 4)$$

$$\begin{bmatrix} \rho \\ u \\ T \end{bmatrix}_i^{t=0} = \begin{bmatrix} 0.395 \\ 1.5 \times \sqrt{0.6897} \\ 0.6897 \end{bmatrix} \qquad 0 \le X_{bar} \le 1$$



Problem ID7: Hoffmann Nozzle Isentropic flow

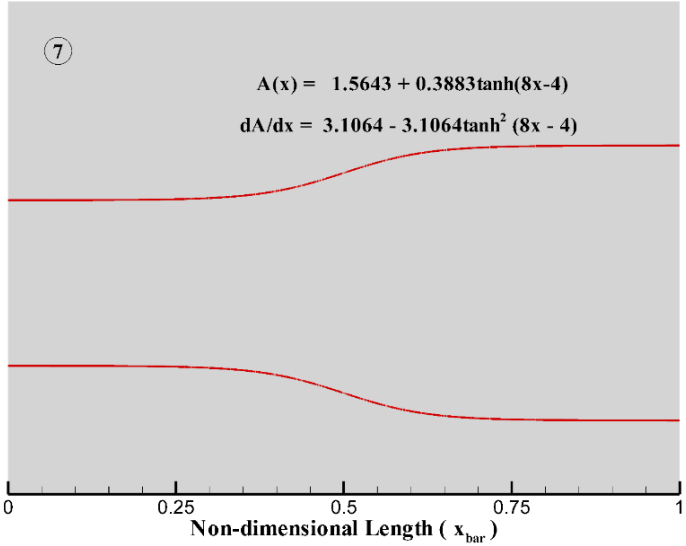
$$A(x) = 1.5643 + 0.3883\tanh(8x - 4)$$

Inflow Conditions

$$\begin{bmatrix} \rho \\ u \\ T \end{bmatrix}_1 = \begin{bmatrix} 0.395 \\ 1.5 \times \sqrt{0.6897} \\ 0.6897 \end{bmatrix}$$

Outflow Conditions

$$\begin{bmatrix} \rho \\ u \\ T \end{bmatrix}_{Imax}^{n} = 2\begin{bmatrix} \rho \\ u \\ T \end{bmatrix}_{Imax-1}^{n} - \begin{bmatrix} \rho \\ u \\ T \end{bmatrix}_{Imax-2}^{n}$$

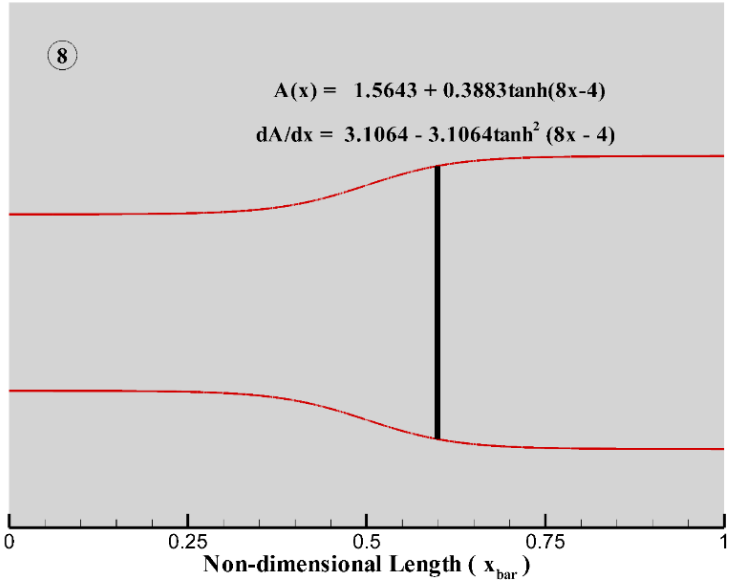


Problem ID8: Hoffmann Nozzle Shock flow

Initial Conditions

$$A(x) = 1.5643 + 0.3883\tanh(8x - 4)$$

$$\begin{bmatrix} \rho \\ u \\ T \end{bmatrix}_i^{t=0} = \begin{bmatrix} 0.395 \\ 1.5 \times \sqrt{0.6897} \\ 0.6897 \end{bmatrix} \qquad 0 \le X_{bar} \le 1$$



Problem ID8: Hoffmann Nozzle Shock flow

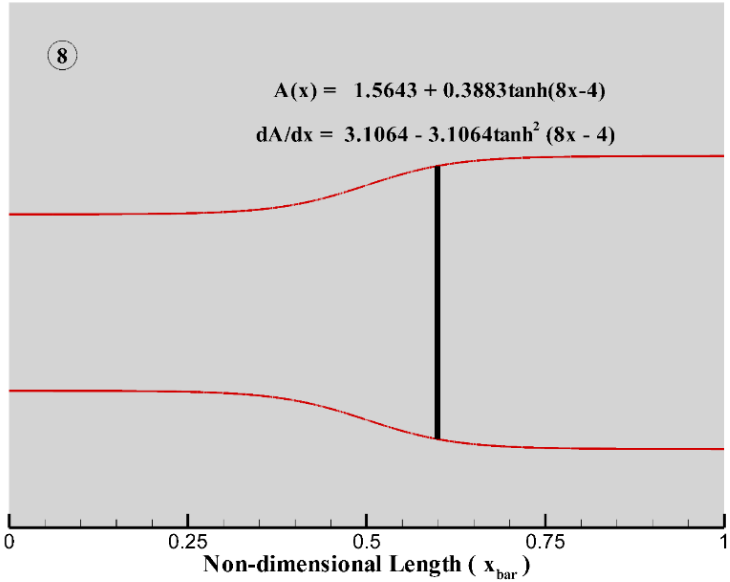
$$A(x) = 1.5643 + 0.3883\tanh(8x - 4)$$

Inflow Conditions

$$\begin{bmatrix} \rho \\ u \\ T \end{bmatrix}_1 = \begin{bmatrix} 0.395 \\ 1.5 \times \sqrt{0.6897} \\ 0.6897 \end{bmatrix}$$

Outflow Conditions

$$\begin{bmatrix} \rho \\ u \\ T \end{bmatrix}_{Imax}^{n} = \begin{bmatrix} \rho_{(i_{max}-1)} \\ 0.5081 \times \sqrt{T_{i_{max}}} \\ \rho_{(i_{max}-1)} \end{bmatrix}$$



Problem ID9 : Feng 2nd Nozzle Isentropic Flow

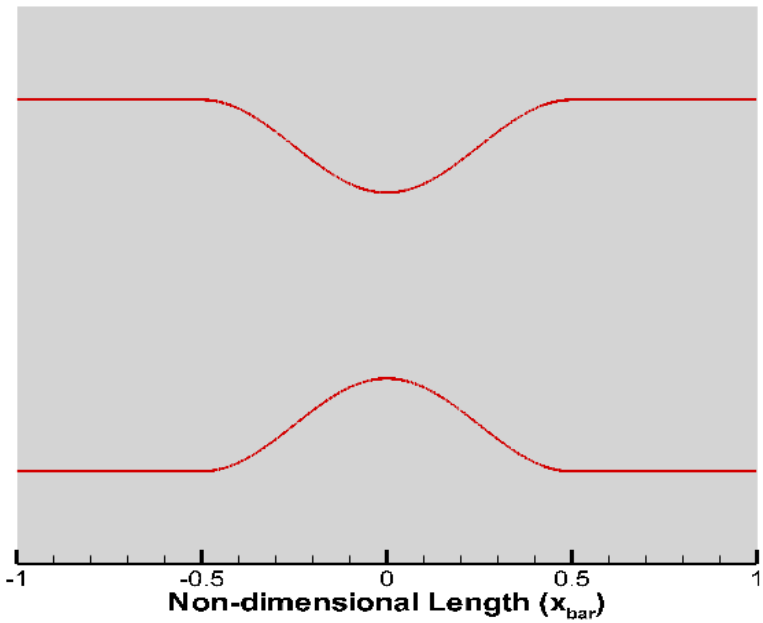
Initial Conditions

$$\begin{bmatrix}\rho\\ u\\ T\end{bmatrix}_{x_{bar}}^{t=0} = \begin{bmatrix}1.0\\ 0.3\\ 1.0\end{bmatrix}_{x_{bar}} \qquad x_{bar} < -0.5$$

$$\begin{bmatrix}\rho\\ u\\ T\end{bmatrix}_{x_{bar}}^{t=0} = \begin{bmatrix}1.0 - 0.366(x_{bar} - 0.5)\\ 0.7\\ 1.0 - 0.167(x_{bar} - 0.5)\end{bmatrix}_{x_{bar}} \qquad -0.5 \le x_{bar} < 0.5$$

$$\begin{bmatrix}\rho\\ u\\ T\end{bmatrix}_{x_{bar}}^{t=0} = \begin{bmatrix}0.634 - 0.3879(x_{bar} - 1.5)\\ 1.5\\ 0.833 - 0.3507(x_{bar} - 1.5)\end{bmatrix}_{x_{bar}} \qquad 0.5 \le x_{bar} \le 1.0$$

$$A(x) = 2.0 \quad for \quad -1.0 \le x < -0.5\,, \qquad 0.5 \le x \le 1.0$$
$$A(x) = 1.0 + sin^2(\pi x) \quad for \quad -0.5 \le x < 0.5\,,$$



Problem ID9: Feng 2nd Nozzle Isentropic Flow

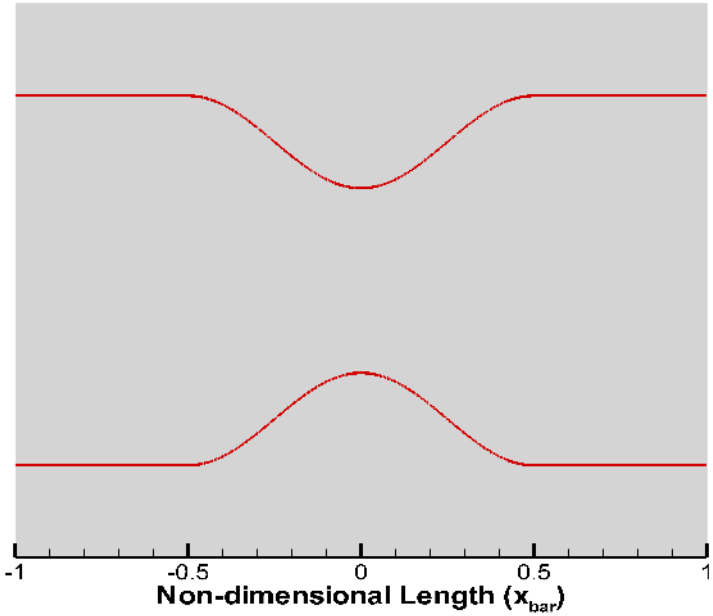
Inflow Conditions

$$\begin{bmatrix}\rho\\ u\\ T\end{bmatrix}_{i=1}^{t} = \begin{bmatrix}1.0\\ 2.0u_2 - u_3\\ 1.0\end{bmatrix}$$

Outflow Conditions

$$\begin{bmatrix}\rho\\ u\\ T\end{bmatrix}_{i_{max}}^{n} = \begin{bmatrix}2.0\rho_{(i_{max}-1)} - \rho_{(i_{max}-2)}\\ 2.0u_{(i_{max}-1)} - u_{(i_{max}-2)}\\ 2.0T_{(i_{max}-1)} - T_{(i_{max}-2)}\end{bmatrix}$$

$$A(x) = 2.0 \quad for \quad -1.0 \le x < -0.5\,, \qquad 0.5 \le x \le 1.0$$
$$A(x) = 1.0 + sin^2(\pi x) \quad for \quad -0.5 \le x < 0.5\,,$$



Problem ID10: Feng 2nd Nozzle Shock Flow

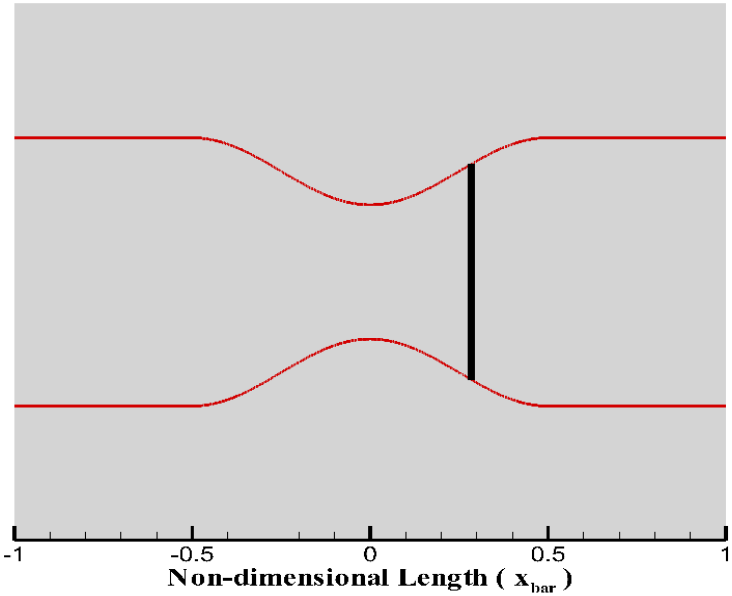
Initial Conditions

$$\begin{bmatrix}\rho\\ u\\ T\end{bmatrix}_{x_{bar}}^{t=0} = \begin{bmatrix}1.0\\ 0.3\\ 1.0\end{bmatrix}_{x_{bar}} \qquad x_{bar} < -0.5$$

$$\begin{bmatrix}\rho\\ u\\ T\end{bmatrix}_{x_{bar}}^{t=0} = \begin{bmatrix}1.0 - 0.366(x_{bar} - 0.5)\\ 0.7\\ 1.0 - 0.167(x_{bar} - 0.5)\end{bmatrix}_{x_{bar}} \qquad -0.5 \le x_{bar} < 0.5$$

$$\begin{bmatrix}\rho\\ u\\ T\end{bmatrix}_{x_{bar}}^{t=0} = \begin{bmatrix}0.634 - 0.3879(x_{bar} - 1.5)\\ 1.5\\ 0.833 - 0.3507(x_{bar} - 1.5)\end{bmatrix}_{x_{bar}} \qquad 0.5 \le x_{bar} \le 1.0$$

$$A(x) = 2.0 \quad for \quad -1.0 \le x < -0.5\,, \qquad 0.5 \le x \le 1.0$$
$$A(x) = 1.0 + sin^2(\pi x) \quad for \quad -0.5 \le x < 0.5$$



Problem ID10: Feng 2nd Nozzle Shock Flow

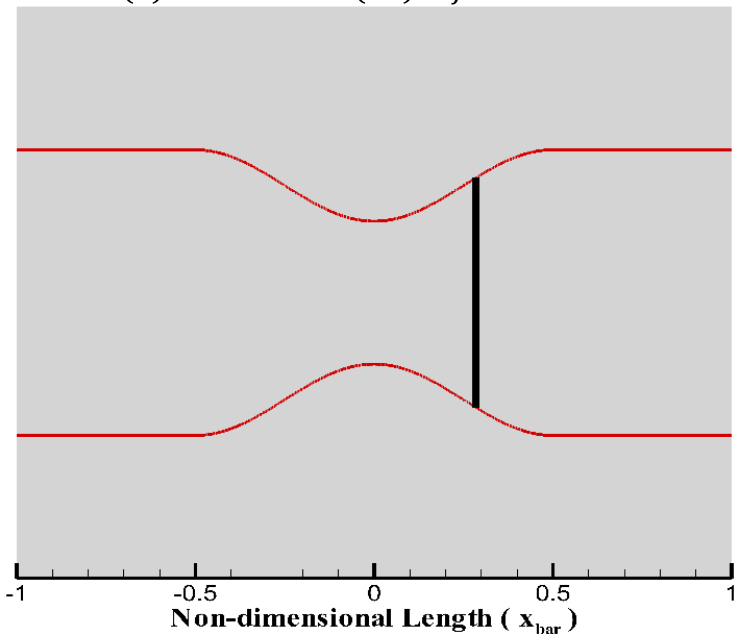
Inflow Conditions

$$\begin{bmatrix} \rho \\ u \\ T \end{bmatrix}_{i=1}^{t} = \begin{bmatrix} 1.0 \\ 2.0u_2 - u_3 \\ 1.0 \end{bmatrix}$$

Outflow Conditions

$$\begin{bmatrix} \rho \\ u \\ T \end{bmatrix}_{i_{max}}^{n} = \begin{bmatrix} \rho_{(i_{max}-1)} \\ 0.43 \\ T_{(i_{max}-1)} \end{bmatrix}$$

$$A(x) = 2.0 \quad for \quad -1.0 \le x < -0.5\,, \qquad 0.5 \le x \le 1.0$$
$$A(x) = 1.0 + sin^2(\pi x) \quad for \quad -0.5 \le x < 0.5\,,$$



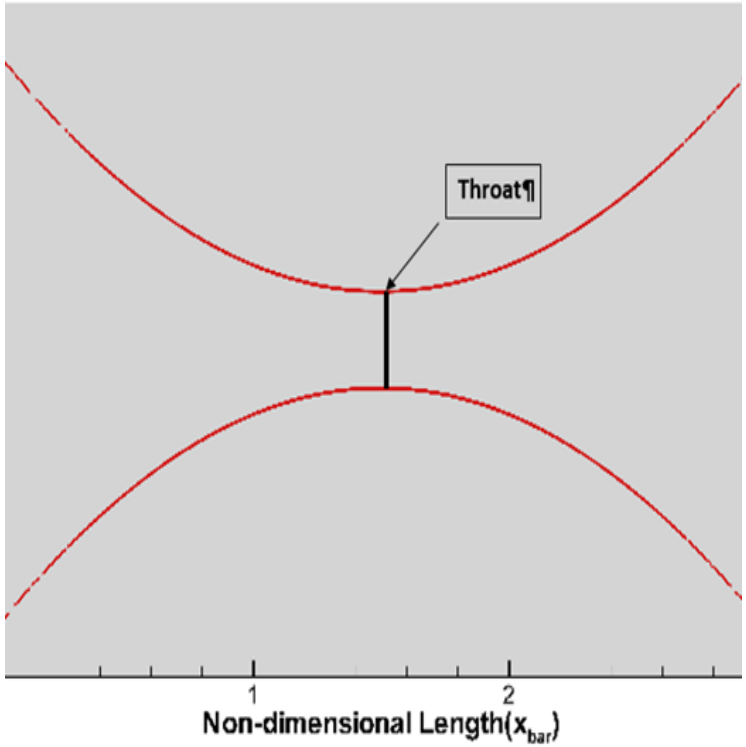
Locations where Exact Solutions are known

- ***Pointwise (Throat)***

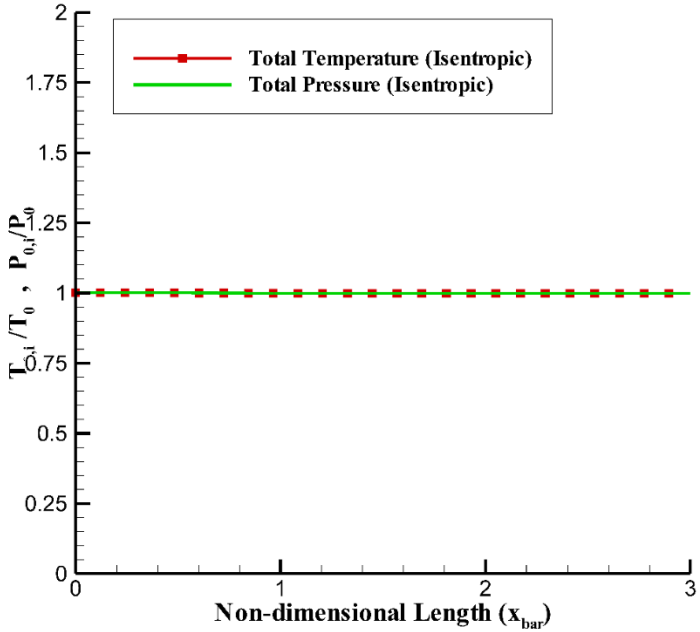
$$T_0/T^*, \qquad P_0/P^*, \qquad M^*$$

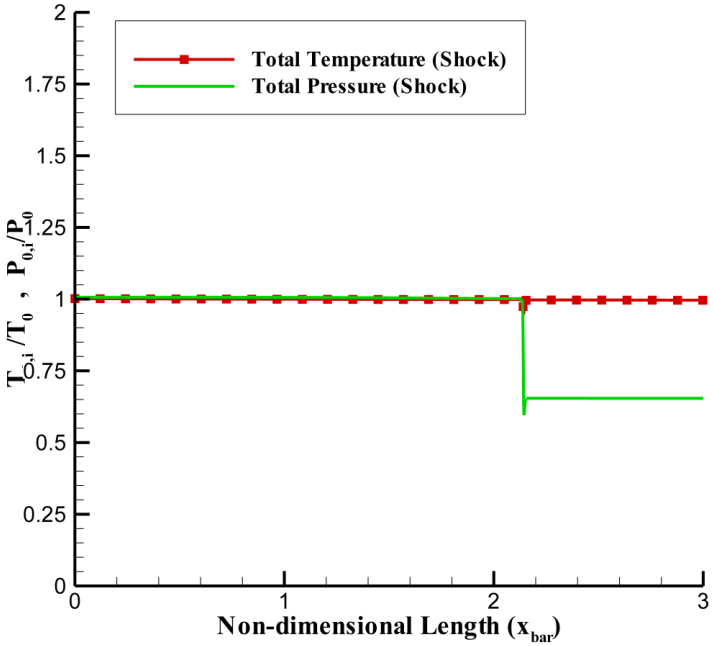
- ***Distributed (Entire Nozzle)***

$$\boldsymbol{T_0}\ is\ constant, \qquad \dot{\boldsymbol{m}}\ is\ constant$$

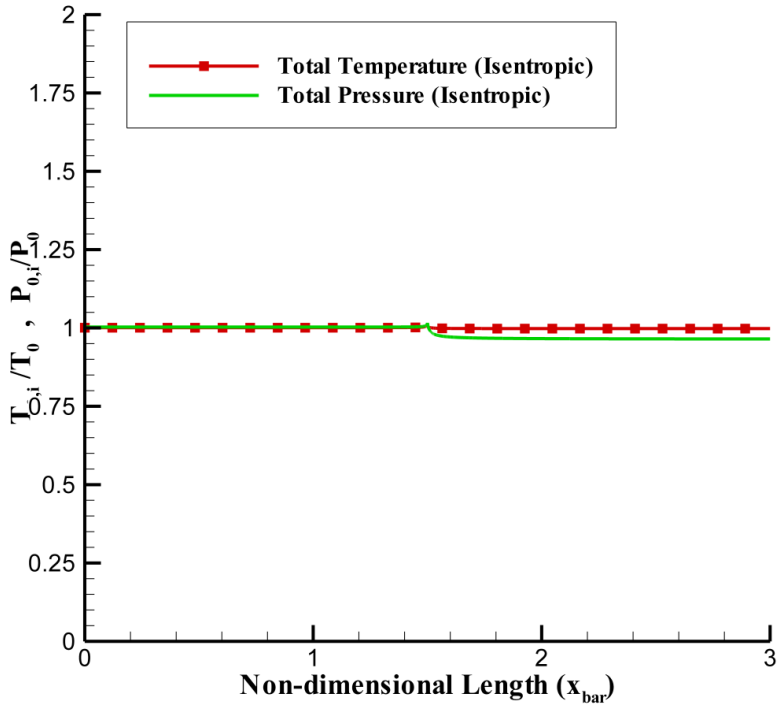


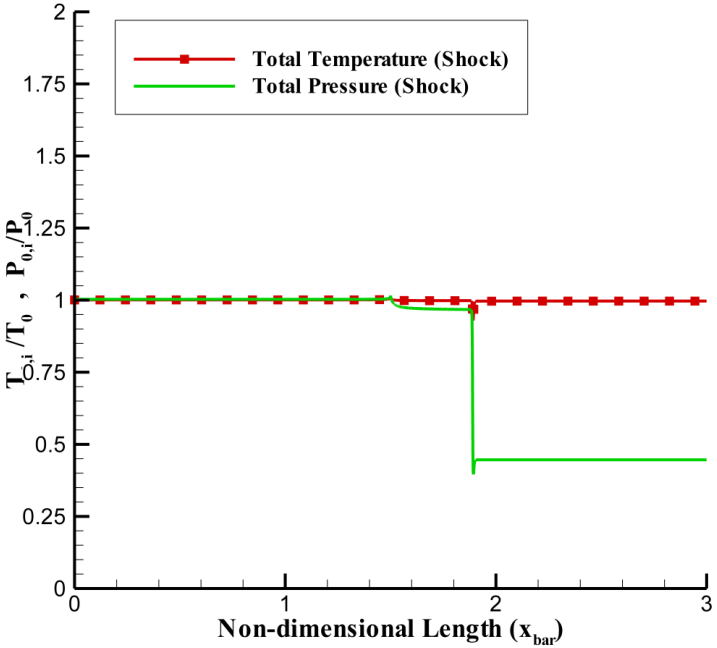
Problem ID1 and ID2



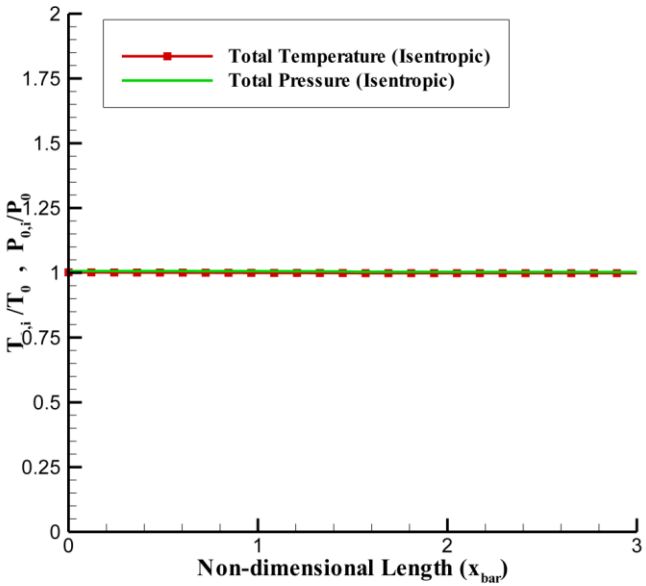


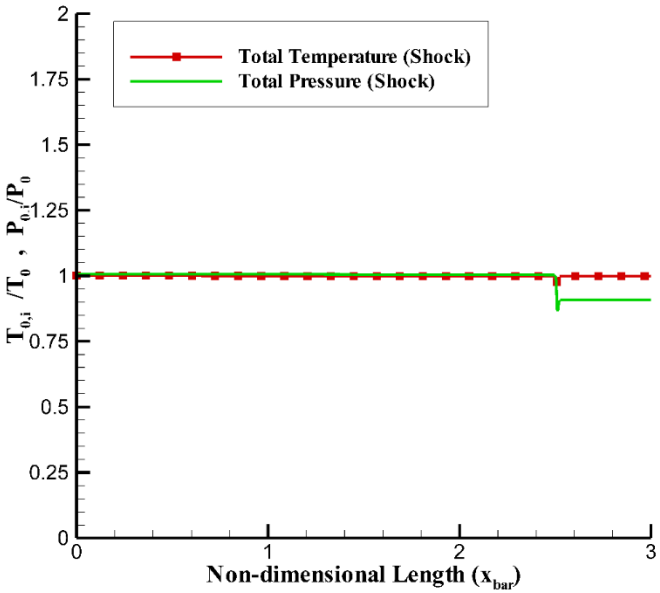
Problem ID3 and ID4



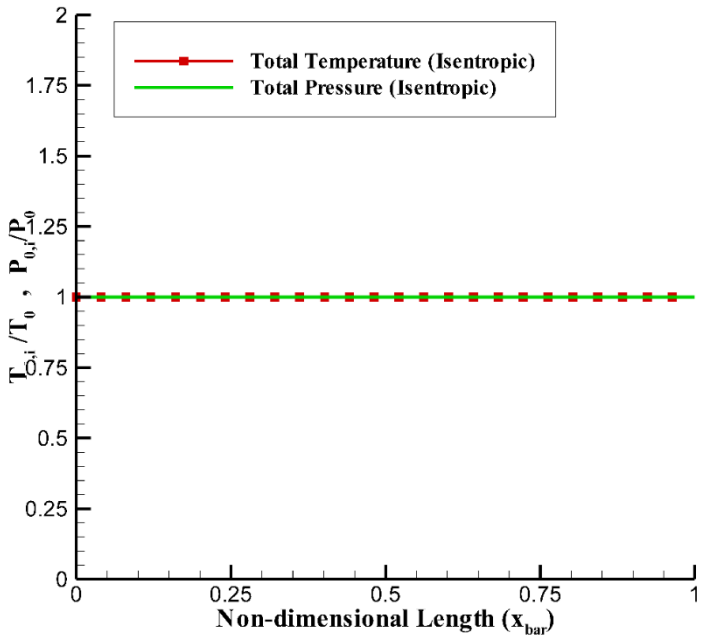


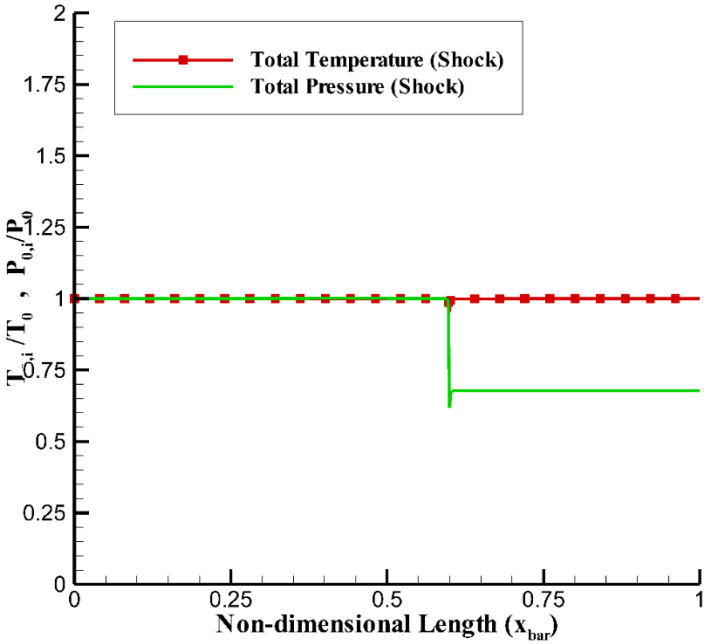
Problems ID5 and ID6



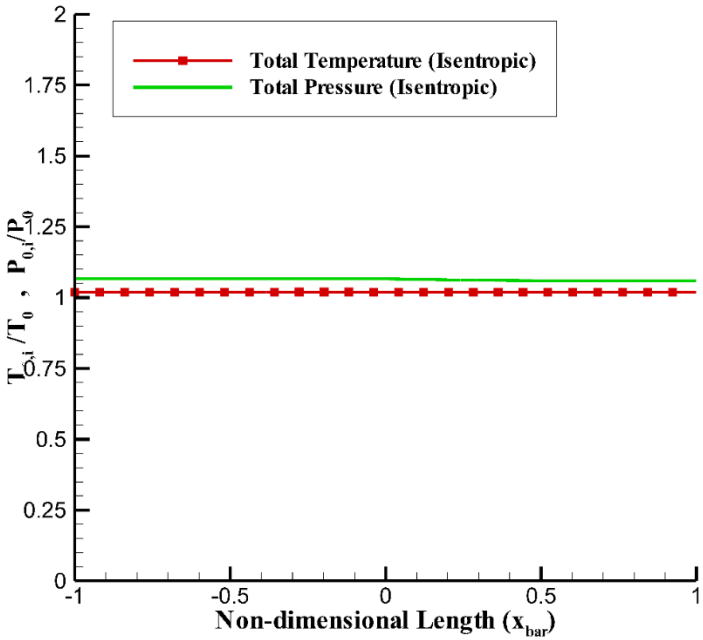


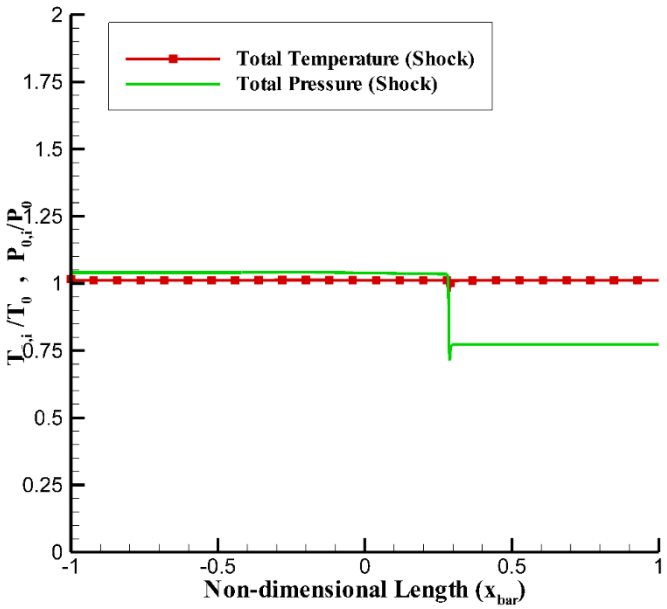
Problem ID7 and ID8

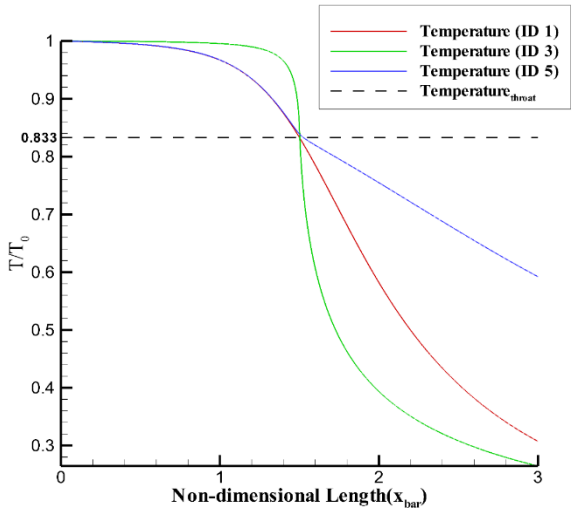


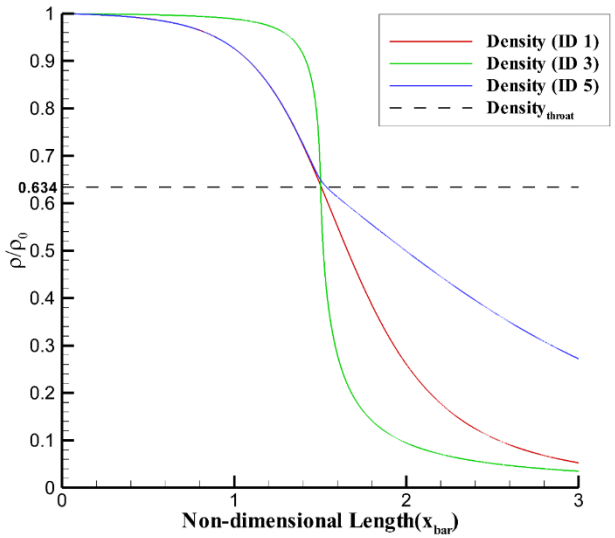


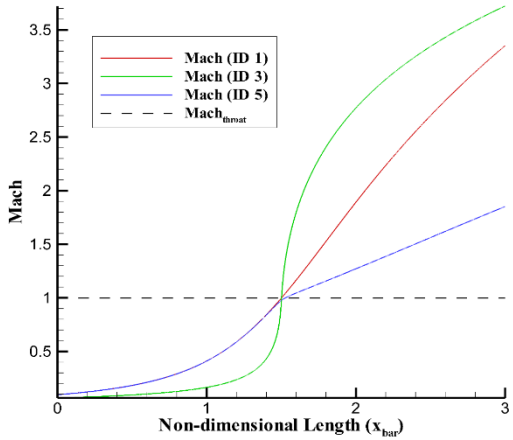
Problem ID9 and ID10



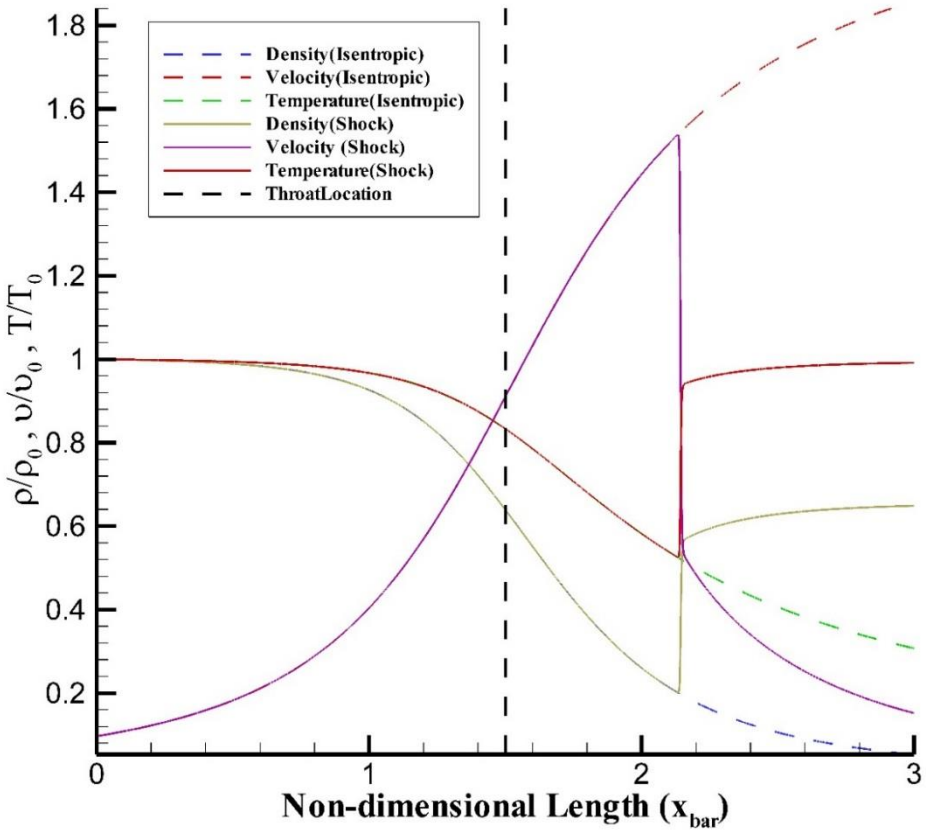



Temperature (ID 1)
Temperature (ID 3)
Temperature (ID 5)
Temperature$_{throat}$
0.833
T/T$_0$
Non-dimensional Length(x$_{bar}$)

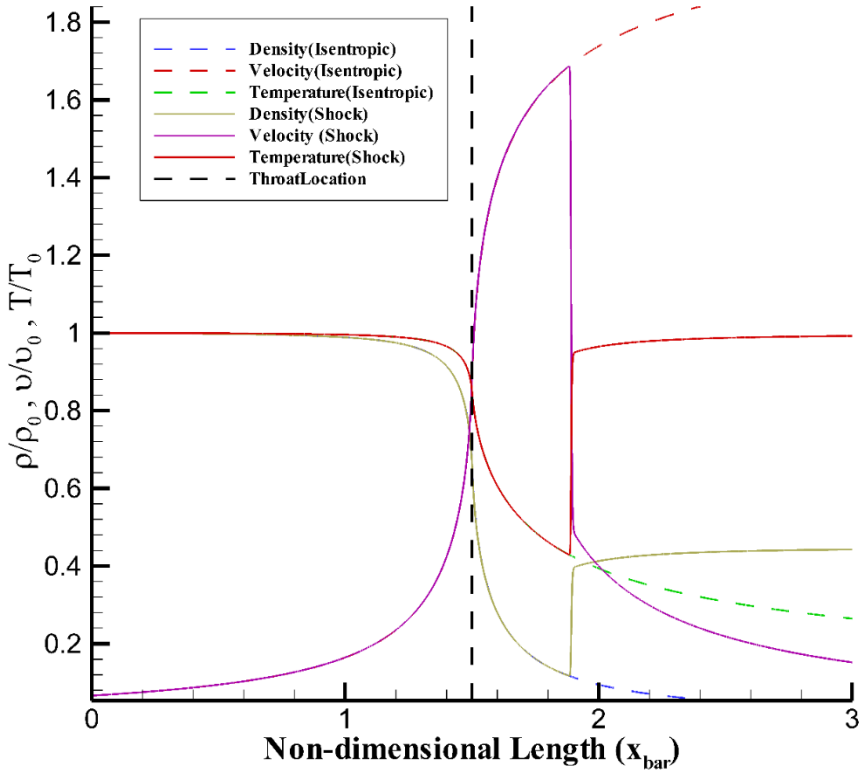

Density (ID 1)
Density (ID 3)
Density (ID 5)
Density$_{throat}$
0.634
ρ/ρ$_0$
Non-dimensional Length(x$_{bar}$)


Mach (ID 1)
Mach (ID 3)
Mach (ID 5)
Mach$_{throat}$
Mach
Non-dimensional Length (x$_{bar}$)

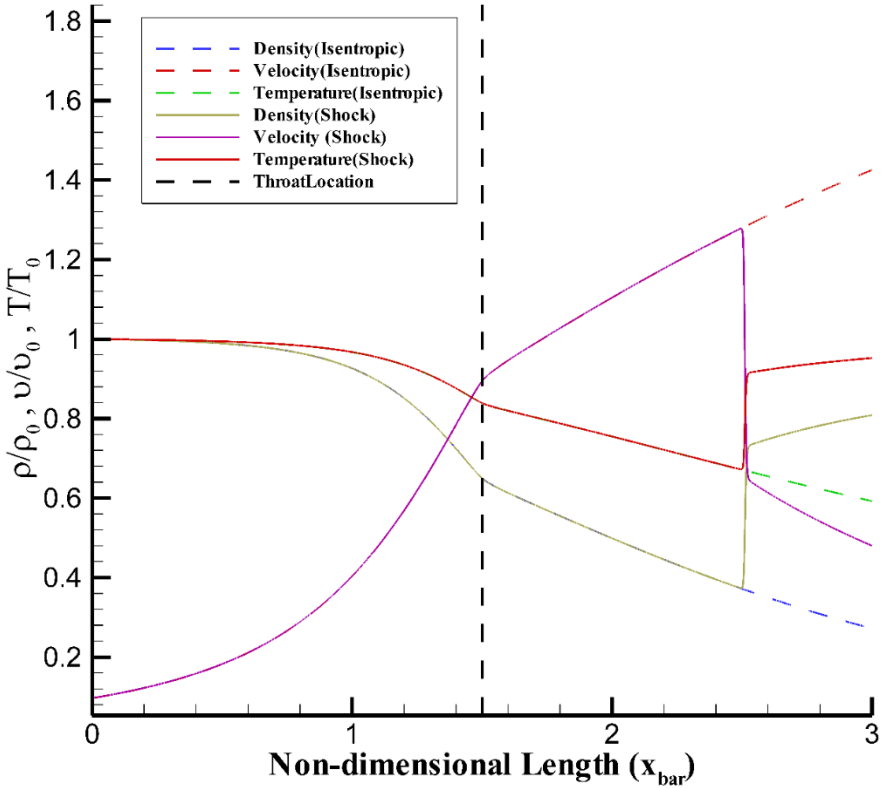
Problem ID1 and ID2



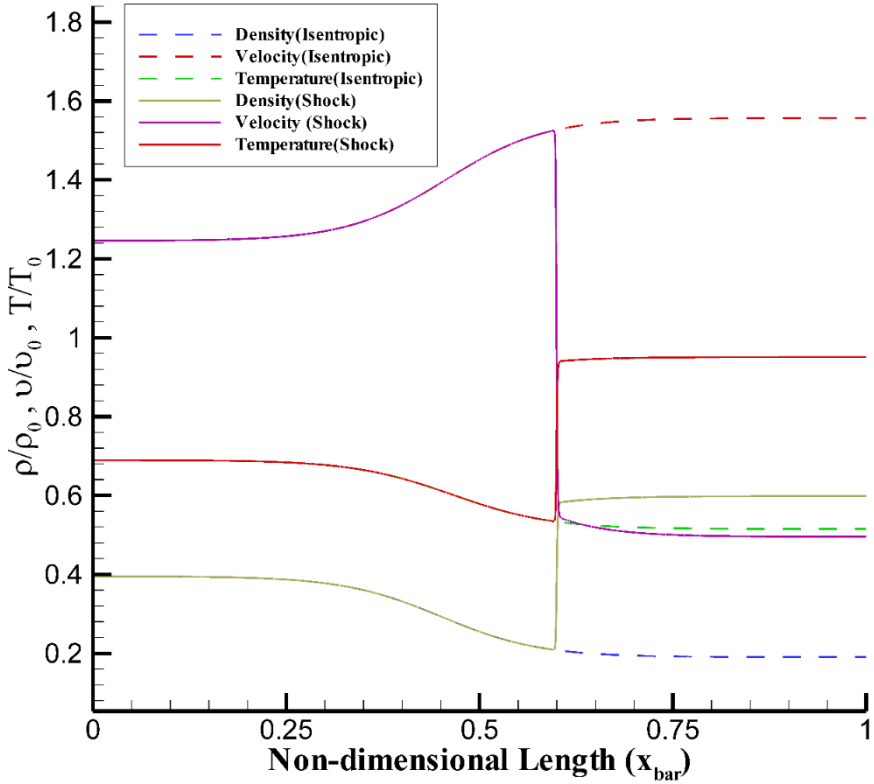
Problem ID3 and ID4



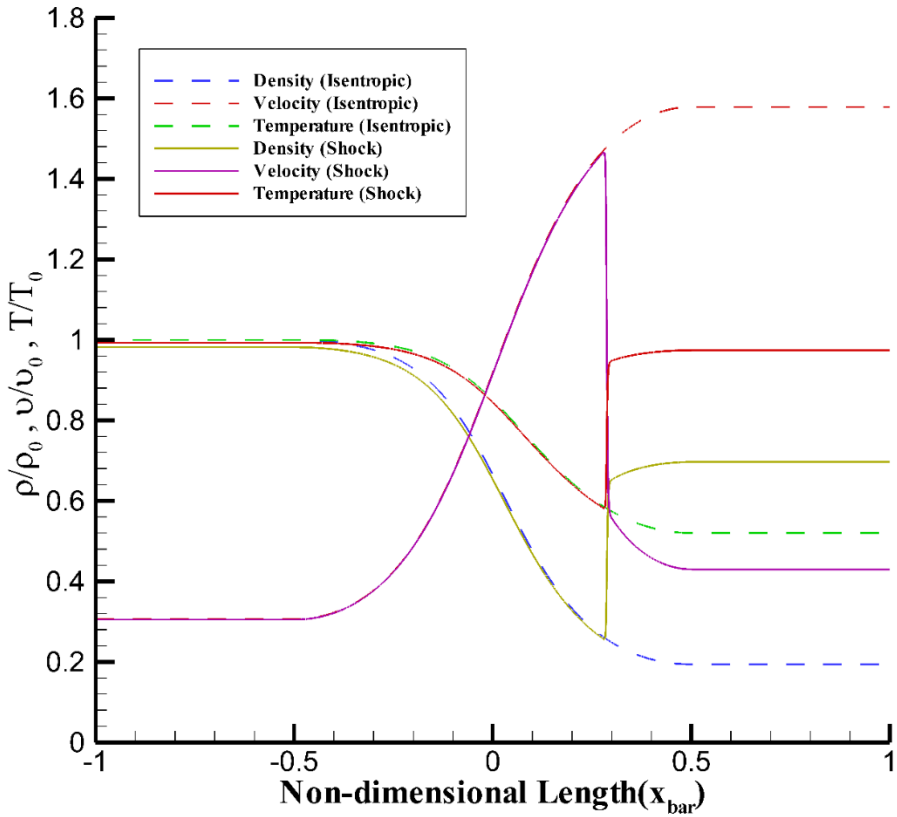
Problem ID5 and ID6



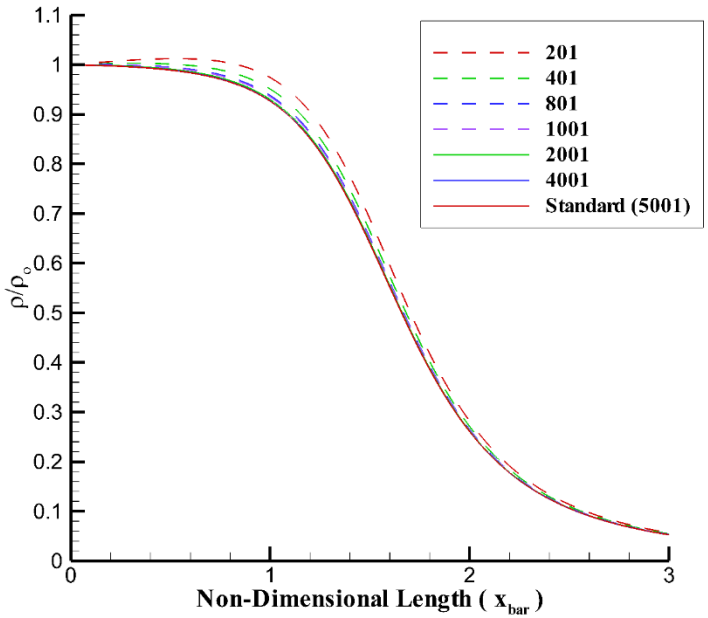
Problem ID7 and ID8



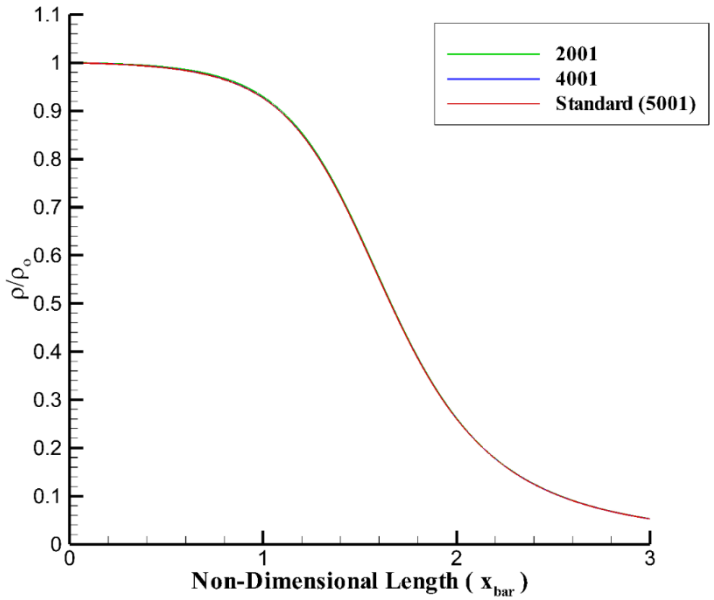
Problem ID9 and ID10



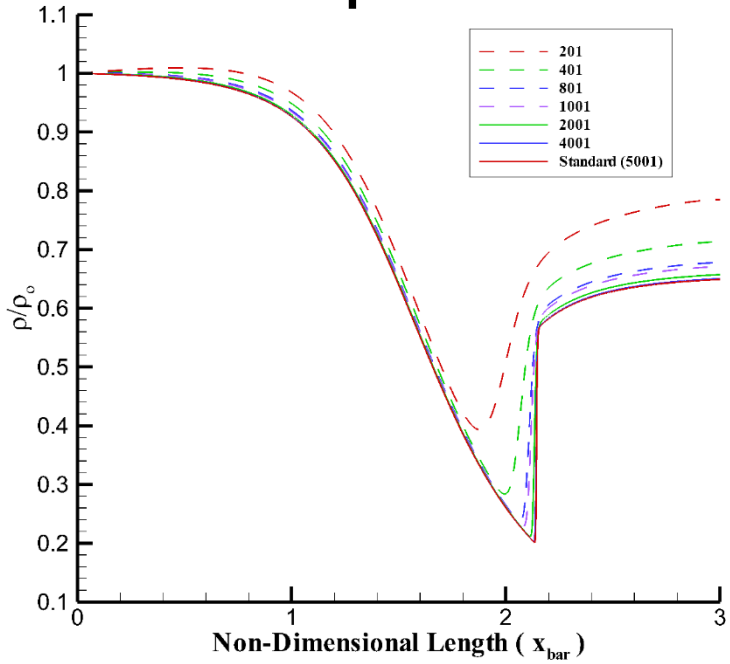
Grid Comparison



Grid Independence



Grid Comparison



Grid Independence

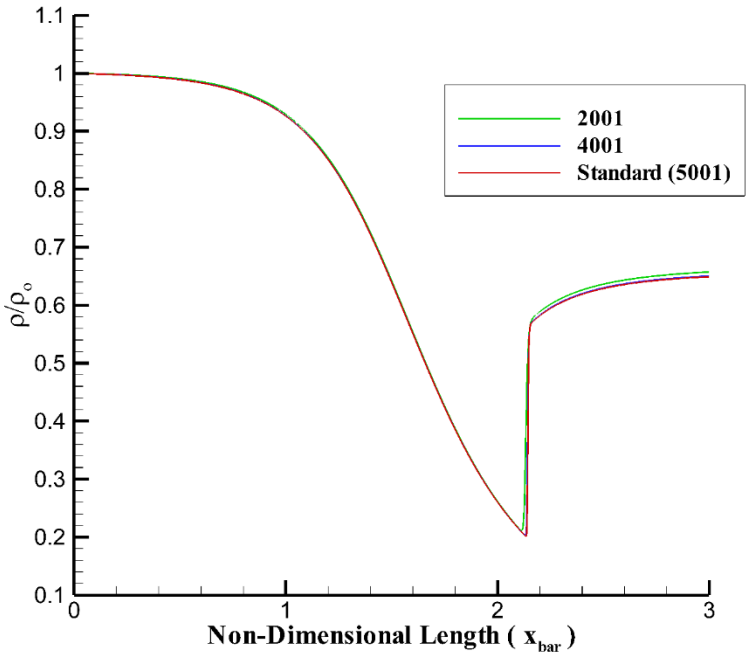


Table 1. Table of Error for Isentropic Problems

Common Assumptions : Numerical Standard Solution (Maximum Grid points : 5001)
Tolerance (10^{-13})

	Point data (Throat Location)			Distributed Data		
	$\varepsilon\left({T_o}/{T_*}\right)\%$	$\varepsilon\left({P_o}/{P_*}\right)\%$	$\varepsilon(M)\ \%$	$\sqrt{\frac{\sum_i^N (1.0-T_0)^2}{\sum_i^N (1.0)^2}}$	$\sqrt{\frac{\sum_i^N (1.0-P_0)^2}{\sum_i^N (1.0)^2}}$	$\sqrt{\frac{\sum_i^N (\dot{m}_{exact}-\dot{m}_i)^2}{\sum_i^N (\dot{m}_{exact})^2}}$
1	1.1548e-02	8.2780e-01	3.2835e-01	1.3137E-003	4.5326E-003	4.7422E-003
3	2.0910	6.7852	4.6841	1.5456E-003	2.377E-002	3.1330E-002
5	6.9240e-01	3.1886	2.4376	1.1432E-003	4.8848E-003	3.9372E-003
9	2.0551	6.6931	0.4363	1.9058E-002	6.4387E-002	5.6816E-002
Max Error	**2.0910**	**6.7852**	**4.6841**	**1.9058E-002**	**6.4387E-002**	**5.6816E-002**

Table 2. Table of Error for Shock Problems

Problem ID	Point data			Distributed Data(L2 Norm)	
	$\varepsilon\left({}^{T_o}/_{T^*}\right)\%$	$\varepsilon\left({}^{P_o}/_{P^*}\right)\%$	$\varepsilon(M)\%$	$\sqrt{\frac{\sum_i^N (1.0 - T_0)^2}{\sum_i^N (1.0)^2}}$	$\sqrt{\frac{\sum_i^N (m_{exact} - m_i)^2}{\sum_i^N (m_{exact})^2}}$
2	9.5686e-04	8.1800e-01	3.0791e-01	2.9257E-003	1.0303E-002
4	2.0571	6.6744	4.6	3.5008E-003	3.4672E-002
6	6.7717e-01	3.0498	2.2890	1.1835E-003	4.5238E-003
8	-	-	-	2.0130E-003	6.5547E-003
10	2.0490	6.6817	0.4241	1.8957E-002	5.7761E-002
Max Error	**2.0571**	**6.6817**	**4.6**	**1.8957E-002**	**5.7761E-002**

CONCLUSIONS

- The results show that the maximum error was 6.78% and occurred at the throat of the nozzle with an acute throat gradient.

- The standard solution can therefore be used in place of an analytical solution for the purpose of error analysis