Implementation of a WENO High-Resolution Scheme

Presented By
Michael F. Harris Ph.D.
Launch Services Program, KSC

Thermal & Fluids Analysis Workshop
TFAWS 2024
August 26-30, 2024
NASA Glenn Research Center
Cleveland, OH

Introduction

- CFD at Launch Services Program
- Why use high order methods?
- Governing Equations: 2-D Euler Equations
- Immersed Boundary Method
- Moving Least Squares and Boundary Conditions
- Time Integration - SSP RK3
- Spatial Discretization – Lax-Friedrichs
- WENO Reconstruction
- Examples
  - Supersonic flow over cylinder
  - Forward facing step
  - Double Mach reflection

CFD at Launch Services Program

- Commercial CFD software used
  - StarCCM+ for general use
- NASA software used
  - Launch Ascent Vehicle Aerodynamics (LAVA)
    - Ignition overpressure and launch acoustic predictions
  - Overflow
    - External flow / aerodynamic predictions of launch vehicles
    - 6 DOF or prescribed motion simulations for booster separation
  - Fun3D
  - Vulcan
    - Hypersonic / Re-entry
    - Commercial launch vehicles are able to recover hardware
- Meshing
  - Pointwise, CGT, StarCCM+

Why use high order methods?

- Advantages
  - Higher accuracy can be achieved on a coarser mesh
    - Much better results compared to lower order schemes on a fine mesh
    - More efficient and less time consuming in that regard
- Disadvantages
  - Instabilities: High order methods like smooth solutions
    - Solution: A switch between high order and low order schemes is needed across shocks or steep gradients $f = (1 - \phi) f_{low} + \phi f_{high}$ where $0 \leq \phi \leq 1$ (TVD schemes, Limiters)
  - Complexities: Difficult to implement on irregular grids and along boundaries
  - Computational expense: Larger stencil, more computational resources
- If interested in high-speed flows and capturing thin shock waves
  - Higher accuracy is needed for effective shock capturing
  - Low order method need many grid points near the shock to resolve it
    - Grid adaption or adaptive mesh refinement (AMR)
  - Or use a higher order method such as WENO

Governing Equations

- 2D Euler Equations for calorically perfect gas

$$\frac{\partial Q}{\partial t} + \frac{\partial F(Q)}{\partial x} + \frac{\partial G(Q)}{\partial y} = 0$$

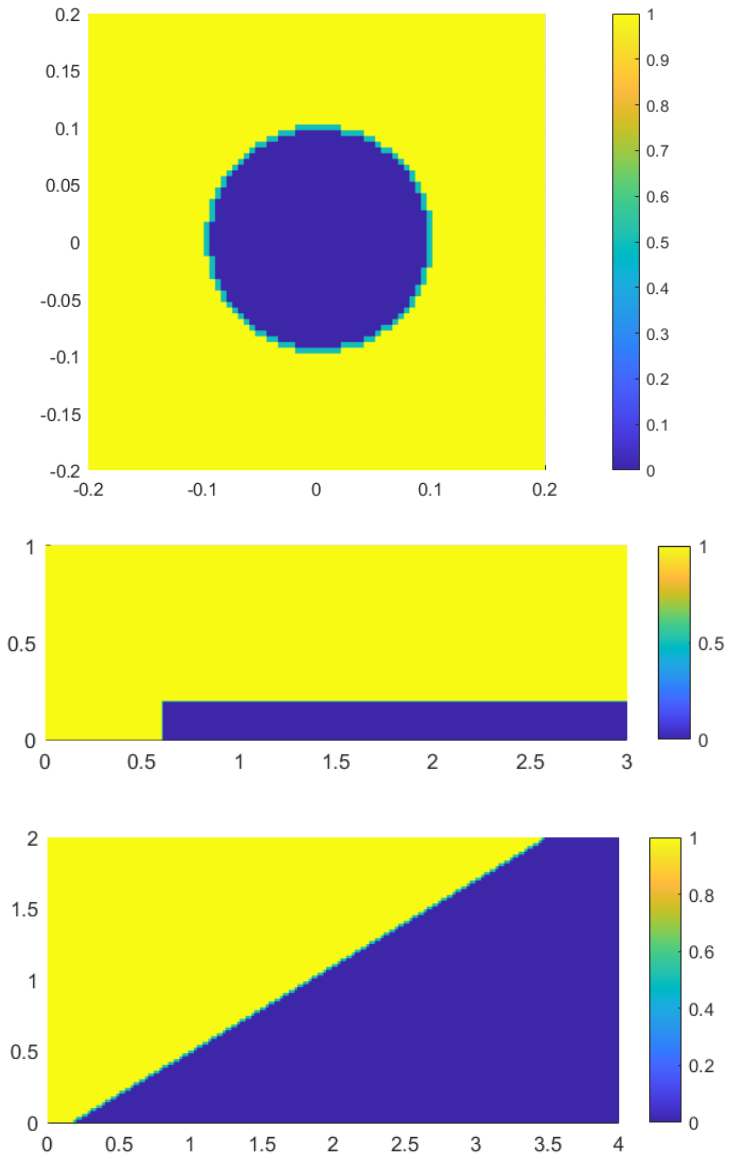
$$Q = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho e_t \end{bmatrix} \qquad F(Q) = \begin{bmatrix} \rho u \\ \rho u^2 + P \\ \rho uv \\ (\rho e_t + P)u \end{bmatrix} \qquad \mathrm{G}(Q) = \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + P \\ (\rho e_t + P)v \end{bmatrix}$$

$$e_t = e + \frac{1}{2}(u^2 + v^2) \qquad e = c_v T \qquad P = \rho RT$$

$$\frac{\partial Q}{\partial t} = -\left(\frac{F_{i+1/2,j} - F_{i-1/2,j}}{\Delta x} + \frac{G_{i,j+1/2} - G_{i,j-1/2}}{\Delta y} \right) = R(Q)$$

Immersed Boundary Method

- Cartesian Mesh
  - Sharp interface immersed boundary approach
  - No inaccuracies due to high aspect ratios or skewness
  - Easy to change geometries for testing numerical methods
- Level set function is created to differentiate between fluid domain, boundary (ghost nodes), and interior grid points (blanked or solid)
- Moving Least Squares (MLS) is used to interpolate inside the fluid domain and apply boundary conditions to ghost nodes
- Inviscid flow- slip flow on walls



Moving Least Squares

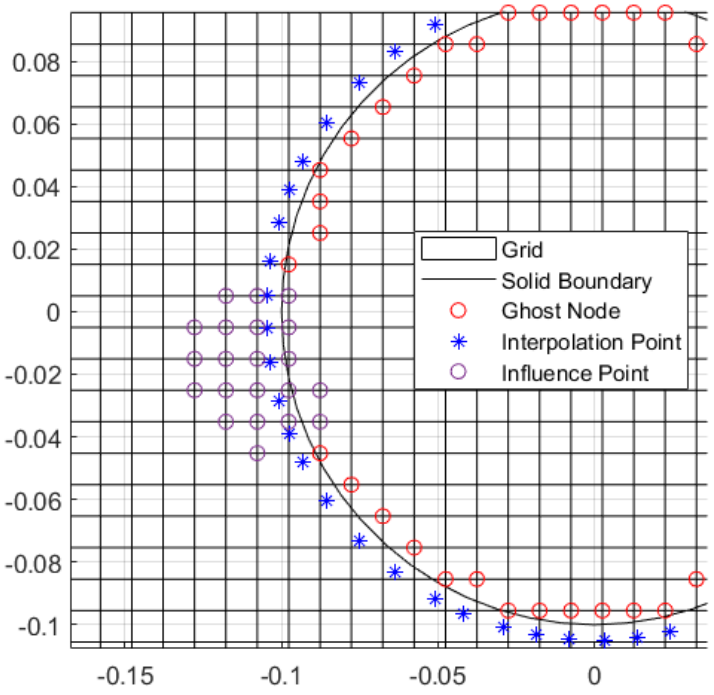
- Moving Least Squares (MLS) used to interpolate field values in the fluid region
  - For a set of chosen polynomials P = [1,x,y]^T and using a sample of influence points (x,y) surrounding the interpolation point, we can approximate the field variables

$$f(x,y) = \sum_{j=1}^{NP} a_j P_j(x,y)$$

  - Least squares minimization will give

$$\sum_{j=1}^{NP} a_j \left[\sum_{k=1}^{NF} P_i(x_k,y_k) P_j(x_k,y_k) \right] = \sum_{k=1}^{NF} P_i(x_k,y_k) f(x_k,y_k)$$

$$[C]_{NP,NP}\{a\}_{NP,1} = [P]_{NP,NF}\{f\}_{NF,1}$$



Moving Least Squares

- Solving for the $\{a\}_{NP,1}$ coefficients

$$\{a\}_{NP,1} = [C]^{-1}_{NP,NP}[P]_{NP,NF}\{f\}_{NF,1}$$

- Recall

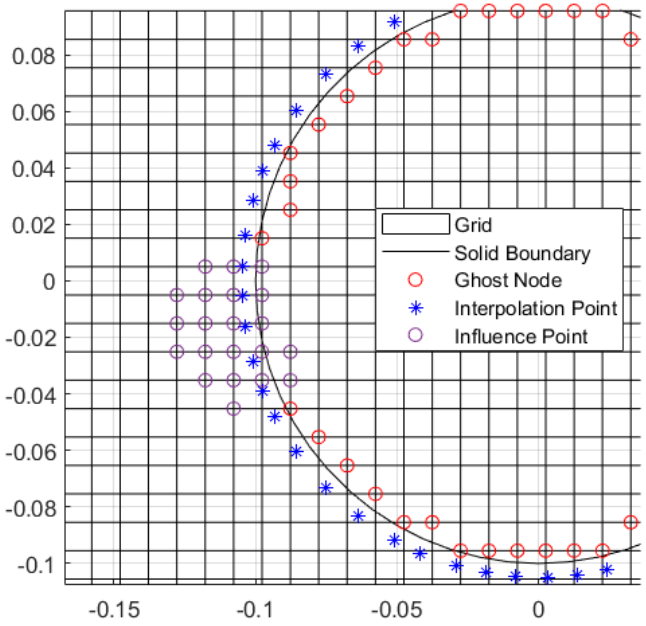
$$f(x,y) = \sum_{j=1}^{NP} a_j P_j(x,y) = \{P\}^T_{1,NP}\{a\}_{NP,1}$$

- After substitution, the field variable can approximated by

$$f(x,y) = \{P\}^T_{1,NP}[C]^{-1}_{NP,NP}[P]_{NP,NF}\{f\}_{NF,1}$$

$$f(x,y) = \{\chi\}^T_{1,NF}\{f\}_{NF,1}$$

- $\{\chi\}_{1,NF}$ can be preprocessed, saved, and used to evaluate the field variables for each interpolation point



Update ghost nodes to enforce velocity tangent to walls or the normal velocity to zero

Time Integration

- Strong Stability Preserving Runge Kutta (SSP RK3) is used for time integration
- Explicit 3 stage scheme

$$\frac{\partial Q}{\partial t} = -\left(\frac{F_{i+1/2,j} - F_{i-1/2,j}}{\Delta x} + \frac{G_{i,j+1/2} - G_{i,j-1/2}}{\Delta y}\right) = R(Q)$$

$$Q_{i,j}^{(1)} = Q_{i,j}^{n} + \Delta t R(Q^{n})$$

$$Q_{i,j}^{(2)} = \frac{3}{4} Q_{i,j}^{n} + \frac{1}{4}\left(Q_{i,j}^{(1)} + \Delta t R(Q^{(1)})\right)$$

$$Q_{i,j}^{n+1} = \frac{1}{3} Q_{i,j}^{n} + \frac{2}{3}\left(Q_{i,j}^{(2)} + \Delta t R(Q^{(2)})\right)$$

Spatial Discretization

- Local Lax-Friedrichs / Rusanov Flux Splitting

$$\frac{\partial Q}{\partial t} = -\left(\frac{F_{i+1/2,j} - F_{i-1/2,j}}{\Delta x} + \frac{G_{i,j+1/2} - G_{i,j-1/2}}{\Delta y}\right)$$

$$F_{i+1/2,j} = \frac{1}{2}\left(F^L_{i+1/2,j} + F^R_{i+1/2,j}\right) - \frac{1}{2}S_{i+1/2,j}\left(Q^R_{i+1/2,j} - Q^L_{i+1/2,j}\right)$$

$$F_{i-1/2,j} = \frac{1}{2}\left(F^L_{i-1/2,j} + F^R_{i-1/2,j}\right) - \frac{1}{2}S_{i-1/2,j}\left(Q^R_{i-1/2,j} - Q^L_{i-1/2,j}\right)$$

$$G_{i,j+1/2} = \frac{1}{2}\left(G^L_{i,j+1/2} + G^R_{i,j+1/2}\right) - \frac{1}{2}S_{i+1/2,j}\left(Q^R_{i,j+1/2} - Q^L_{i,j+1/2}\right)$$

$$G_{i,j-1/2} = \frac{1}{2}\left(G^L_{i,j-1/2} + G^R_{i,j-1/2}\right) - \frac{1}{2}S_{i-1/2,j}\left(Q^R_{i,j-1/2} - Q^L_{i,j-1/2}\right)$$

$$S = max(|\lambda|)$$

S is the maximum of the eigenvalues

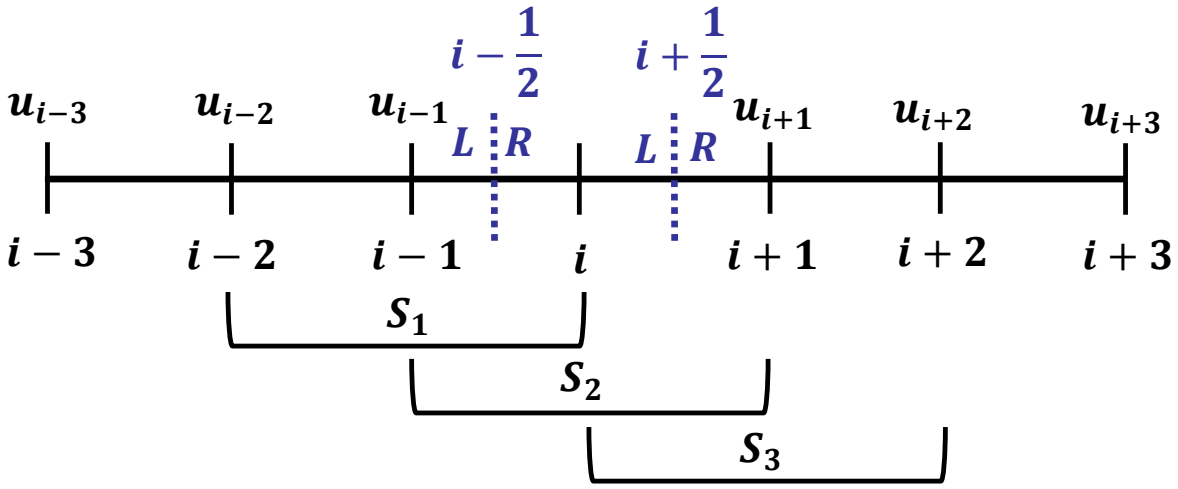
$$\lambda = [u + a, u, u, u - a]$$

Taking a local maximum gives the local Lax-Friedrich flux splitting

Fluxes need to be reconstructed at each cell's left and right interface

WENO Scheme

- Builds upon Essentially Non-Oscillatory (ENO) schemes developed by Harten et. al. [1]
  - ENO reconstruction uses the smoothest stencil among a set of selected stencils that include a given cell
  - Smoothness indicator is used to determine stencil to be used
  - A reconstruction polynomial is built using the single smoothest stencil of k[th] order



$$u_{i+\frac{1}{2}}^{(1)\,L} = \frac{1}{3}u_{i-2} - \frac{7}{6}u_{i-1} + \frac{11}{6}u_i$$

$$u_{i+\frac{1}{2}}^{(2)\,L} = -\frac{1}{6}u_{i-1} + \frac{5}{6}u_i + \frac{1}{3}u_{i+1}$$

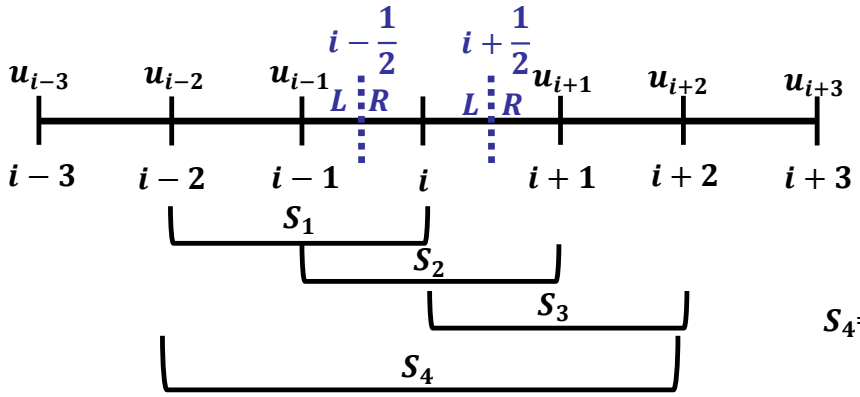
$$u_{i+\frac{1}{2}}^{(3)\,L} = \frac{1}{3}u_i + \frac{5}{6}u_{i+1} - \frac{1}{6}u_{i+2}$$

Third order accurate

$S_1 = [i-2, i-1, i]$ $S_2 = [i-1, i, i+1]$ $S_3 = [i, i+1, i+2]$

WENO Scheme

- Weighted Essentially Non-Oscillatory (WENO) reconstruction uses a convex combination of all the selected stencils (Jiang, Shu) [2]
  - Weights are determined based local smoothness of the stencil such that:
    - Reconstruction polynomial of 2k[th]-1 order is achieved in smooth regions
    - Reverts to k[th] order ENO reconstruction near discontinuities



$S_1 = [i-2, i-1, i]$

$S_2 = [i-1, i, i+1]$

$S_3 = [i, i+1, i+2]$

$S_4 = [i-2, i-1, i, i+1, i+2]$

$$\beta_1 = \frac{13}{12}(u_{i-2} - 2u_{i-1} + u_i)^2 + \frac{1}{4}(u_{i-2} - 4u_{i-1} + 3u_i)^2$$

$$\beta_2 = \frac{13}{12}(u_{i-1} - 2u_i + u_{j+1})^2 + \frac{1}{4}(u_{i+1} - u_{i-1})^2$$

$$\beta_3 = \frac{13}{12}(u_i - 2u_{i+1} + u_{i+2})^2 + \frac{1}{4}(3u_i - 4u_i + u_{i+2})^2$$

$$\alpha_k = \frac{\gamma_k}{(\beta_k + \varepsilon)^2}$$ where ε is small to avoid division by zero

$$\gamma_1 = \frac{1}{10} \quad \gamma_2 = \frac{3}{5} \quad \gamma_3 = \frac{3}{10}$$

$$w_k = \frac{\alpha_k}{\sum_{k=1}^{3} \alpha_k}$$

$$u_{i+\frac{1}{2}}^{(1)\,L} = \frac{1}{3}u_{i-2} - \frac{7}{6}u_{i-1} + \frac{11}{6}u_i$$

$$u_{i+\frac{1}{2}}^{(2)\,L} = -\frac{1}{6}u_{i-1} + \frac{5}{6}u_i + \frac{1}{3}u_{i+1}$$

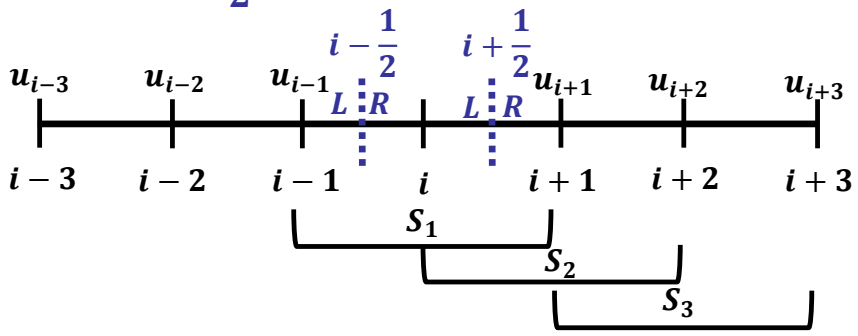
$$u_{i+\frac{1}{2}}^{(3)\,L} = \frac{1}{3}u_i + \frac{5}{6}u_{i+1} - \frac{1}{6}u_{i+2}$$

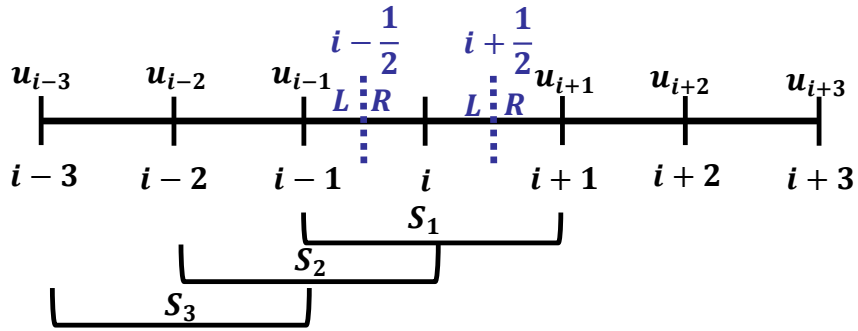
$$u_{i+\frac{1}{2}}^{L} = w_1 u_{i+\frac{1}{2}}^{(1)\,L} + w_2 u_{i+\frac{1}{2}}^{(2)\,L} + w_3 u_{i+\frac{1}{2}}^{(3)\,L}$$

5[th] order accurate if all stencils are smooth

WENO Scheme

- L reconstruction will have 1 more point left of the interface
- R reconstruction will have 1 more point right of the interface
- For $u_{i+\frac{1}{2}}^{(k)\,R}$ the stencil needs to be shifted one cell to the right
- For $u_{i-\frac{1}{2}}^{(k)\,L}$ the stencil needs to be shifted one cell to the left





$$u_{i+\frac{1}{2}}^{(1)\,R} = -\frac{1}{6}u_{i-1} + \frac{5}{6}u_i + \frac{1}{3}u_{i+1}$$

$$u_{i+\frac{1}{2}}^{(2)\,R} = \frac{1}{3}u_i + \frac{5}{6}u_{i+1} - \frac{1}{6}u_{i+2}$$

$$u_{i+\frac{1}{2}}^{(3)\,R} = \frac{1}{3}u_{i+1} - \frac{7}{6}u_{i+2} + \frac{11}{6}u_{i+3}$$

$$u_{i-\frac{1}{2}}^{(1)\,L} = -\frac{1}{6}u_{i+1} + \frac{5}{6}u_i + \frac{1}{3}u_{i-1}$$

$$u_{i-\frac{1}{2}}^{(2)\,L} = \frac{1}{3}u_i + \frac{5}{6}u_{i-1} - \frac{1}{6}u_{i-2}$$

$$u_{i-\frac{1}{2}}^{(3)\,L} = \frac{1}{3}u_{i-1} - \frac{7}{6}u_{i-2} + \frac{11}{6}u_{i-3}$$

WENO Scheme

- In literature, the coefficients for the reconstruction polynomials may be presented differently
  - Finding a Lagrange interpolating polynomial for each stencil and evaluating at i+1/2 for uniform grid spacing, we can obtain

$$u_{i+\frac{1}{2}}^{(1)\,L} = \frac{3}{8}u_{i-2} - \frac{5}{8}u_{i-1} + \frac{15}{8}u_i \qquad u_{i+\frac{1}{2}}^{(2)\,L} = -\frac{1}{8}u_{i-1} + \frac{3}{4}u_i + \frac{3}{8}u_{i+1} \qquad u_{i+\frac{1}{2}}^{(3)\,L} = \frac{3}{8}u_i + \frac{3}{4}u_{i+1} - \frac{1}{8}u_{i+2}$$

with $\gamma_1 = \frac{1}{16}$, $\gamma_2 = \frac{5}{8}$ and $\gamma_3 = \frac{5}{16}$

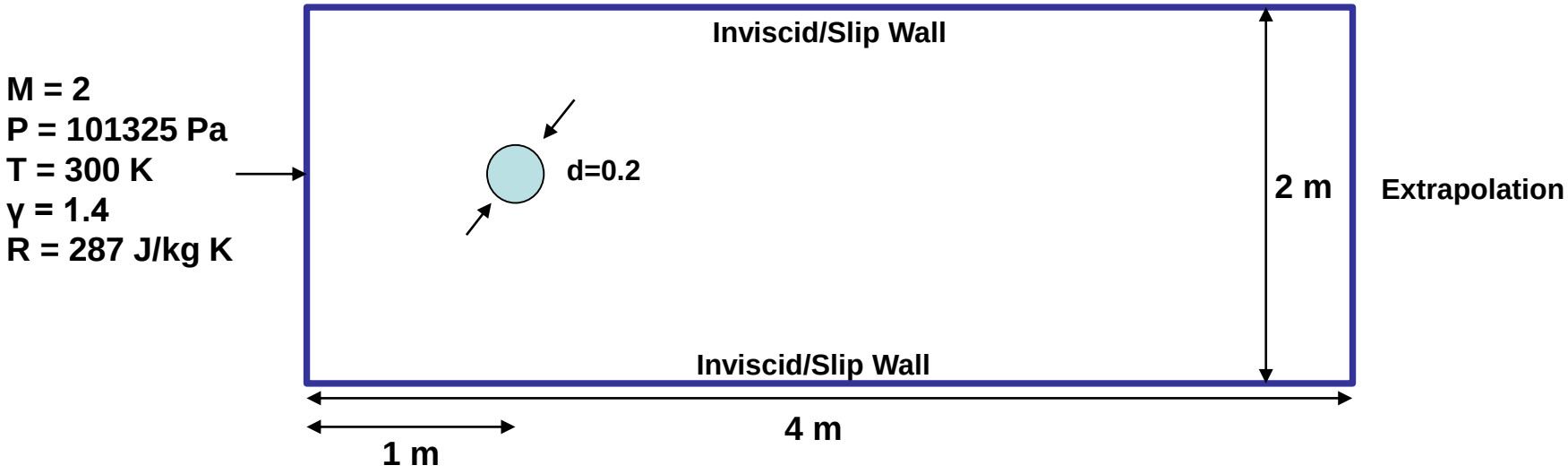
  - Taking the quadratic $P(s) = as^2 + bs + c$ where $s = (x - x_i)$ and averaging over the cell, we can find a, b, and c to satisfy

$$\bar{u}_{i-1} = \int_{-\frac{3\Delta x}{2}}^{-\frac{\Delta x}{2}} P(s)ds \qquad \bar{u}_i = \int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} P(s)ds \qquad \bar{u}_{i+1} = \int_{\frac{\Delta x}{2}}^{\frac{3\Delta x}{2}} P(s)ds$$

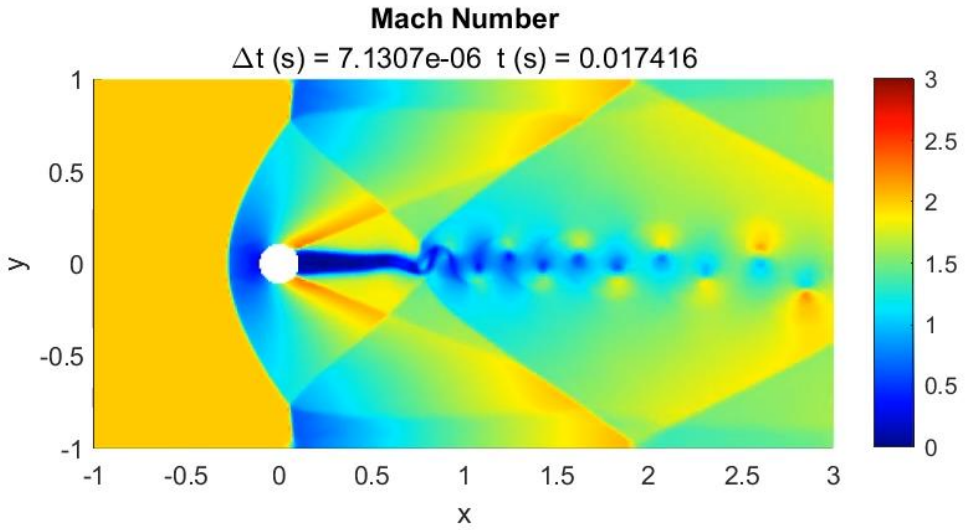
$$\bar{u}_{i+\frac{1}{2}}^{(1)\,L} = \frac{1}{3}u_{i-2} - \frac{7}{6}u_{i-1} + \frac{11}{6}u_i \qquad \bar{u}_{i+\frac{1}{2}}^{(2)\,L} = -\frac{1}{6}u_{i-1} + \frac{5}{6}u_i + \frac{1}{3}u_{i+1} \qquad \bar{u}_{i+\frac{1}{2}}^{(3)\,L} = \frac{1}{3}u_i + \frac{5}{6}u_{i+1} - \frac{1}{6}u_{i+2}$$

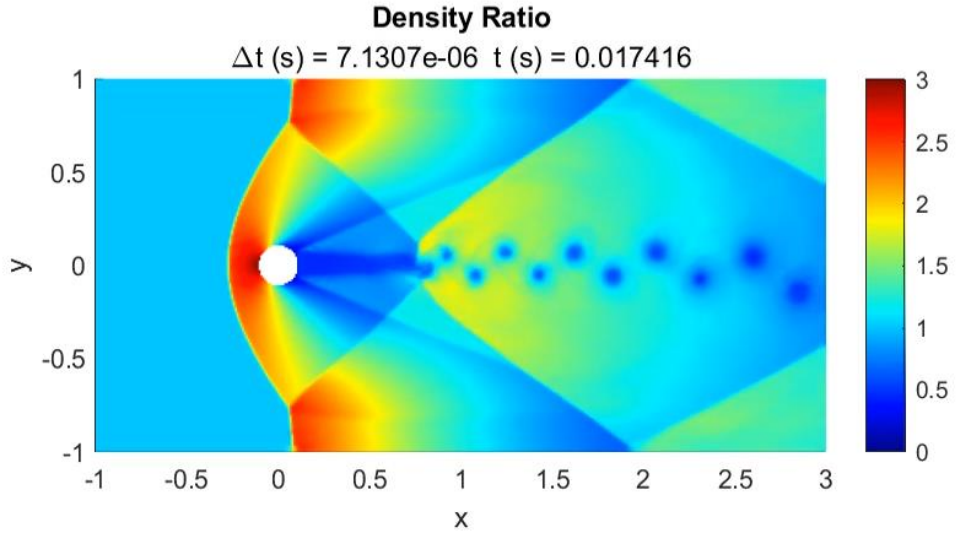
Supersonic Flow over Cylinder

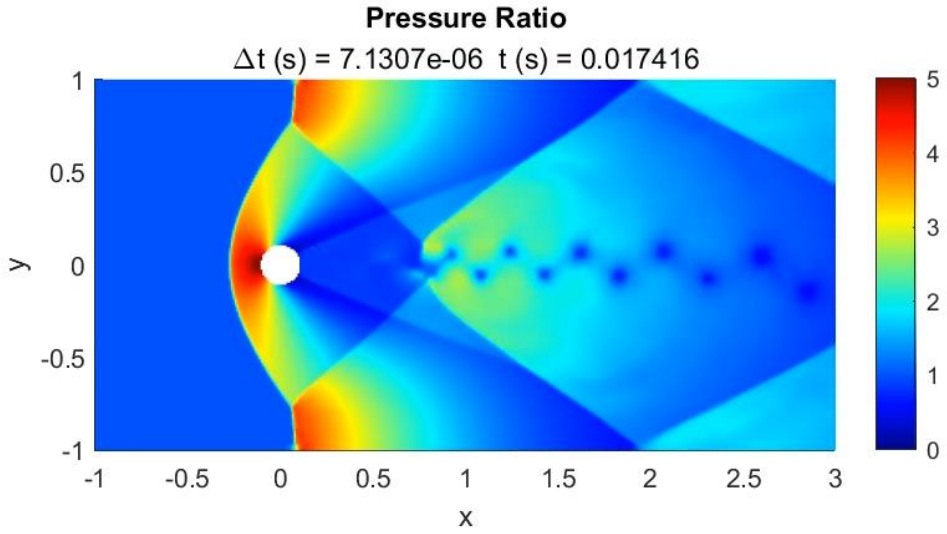
- Test Problem 1
- Domain initialized using inlet conditions
- 400x200 grid points
- 800x400 grid points



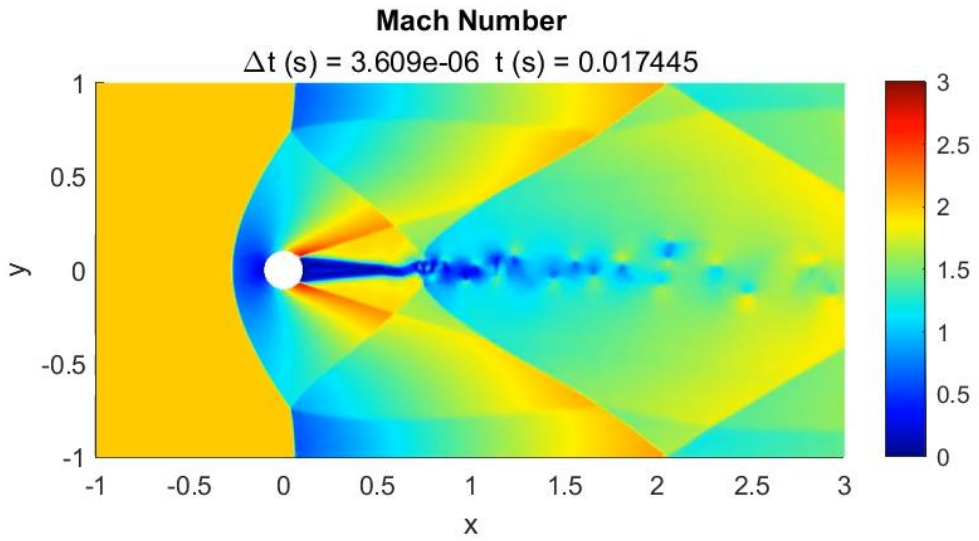
Supersonic Flow over Cylinder (400x200 Grid)

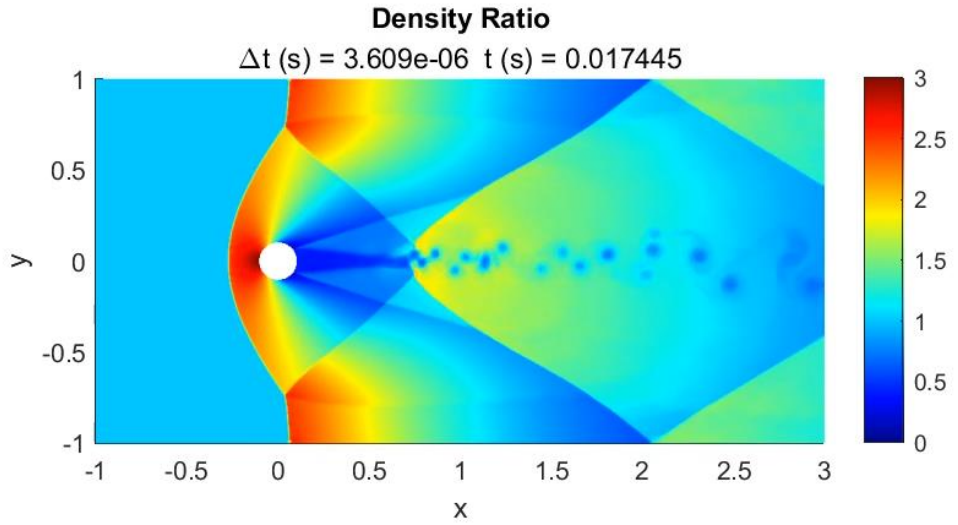


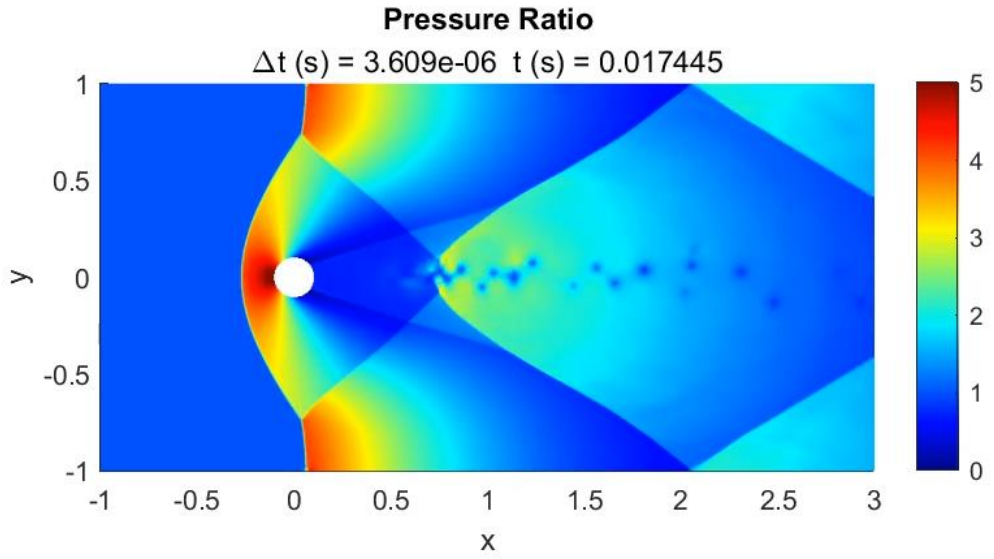




Supersonic Flow over Cylinder (800x400 Grid)

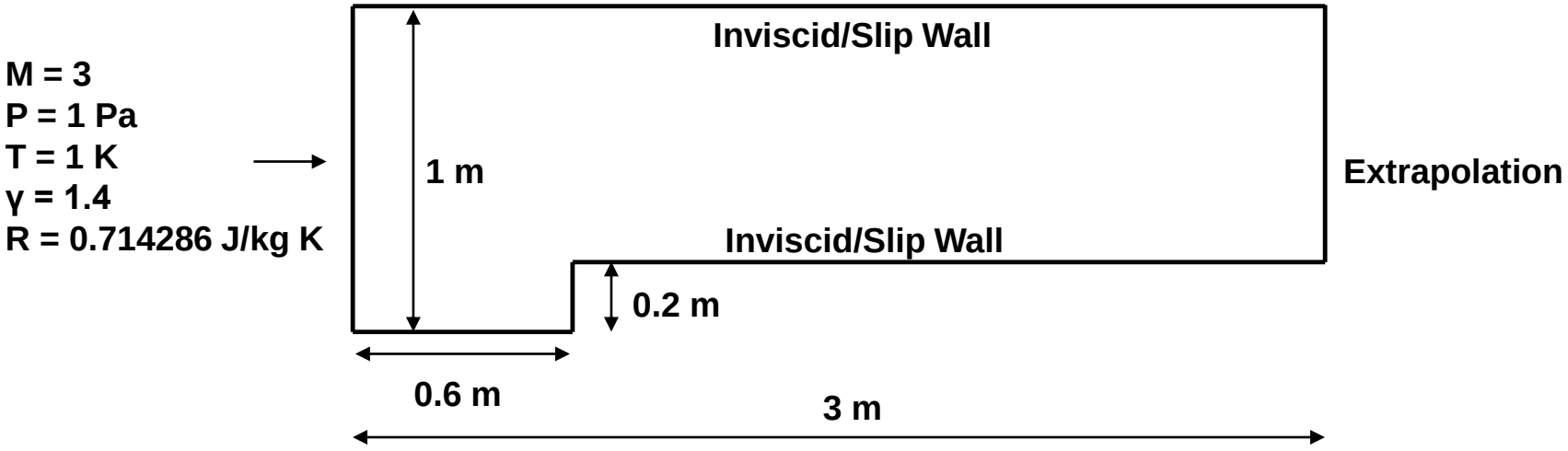




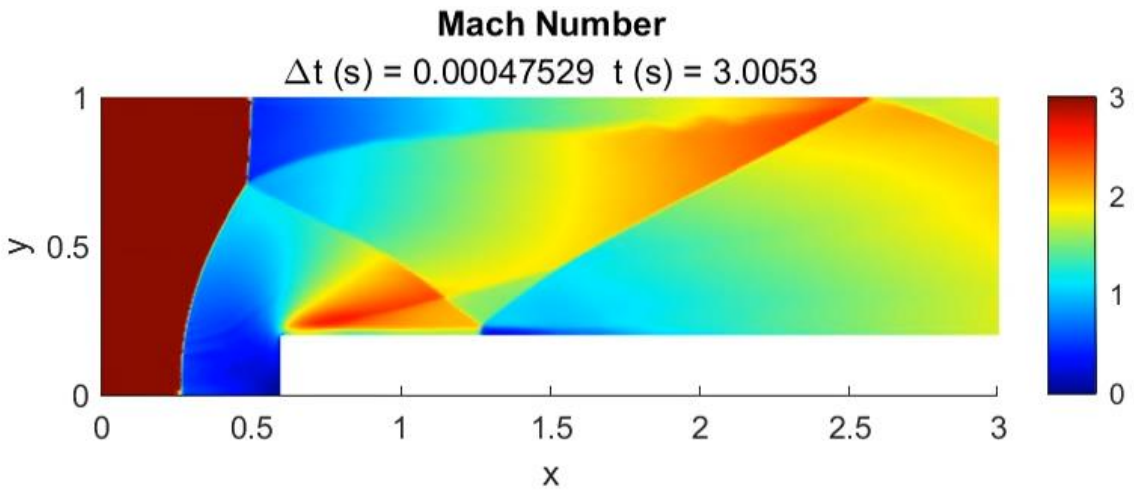


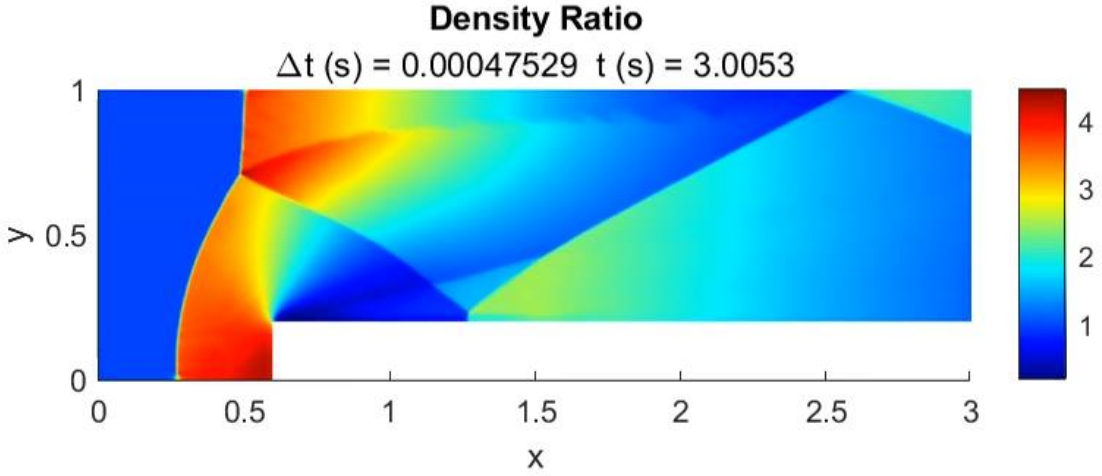
Forward Facing Step

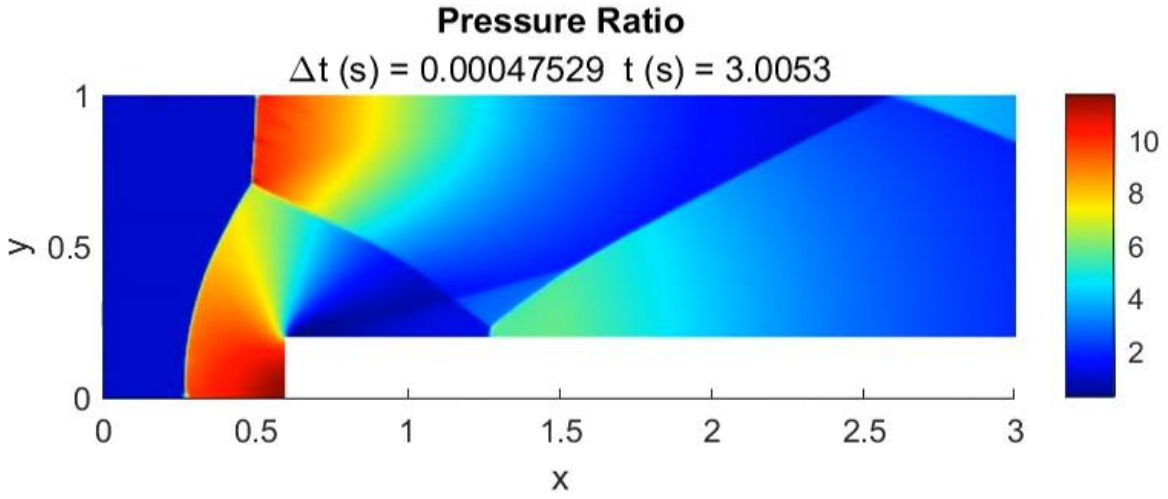
- Test Problem 2
- Domain initialized using inlet conditions
- 600 x 200 grid points



Forward Facing Step (600 x 200 Grid)

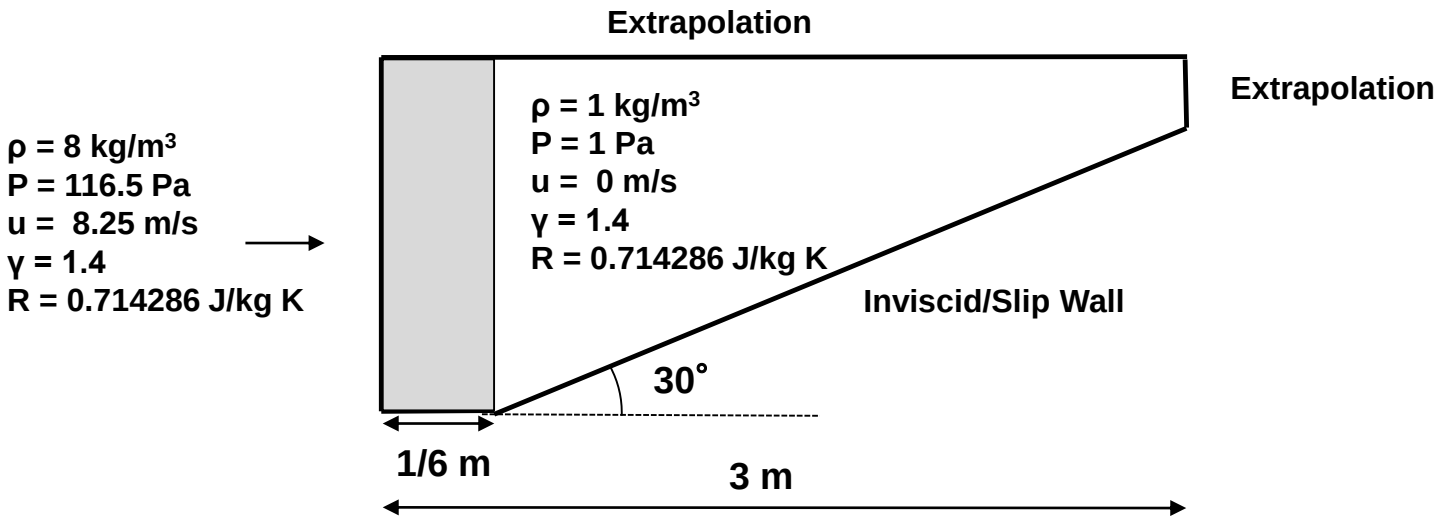






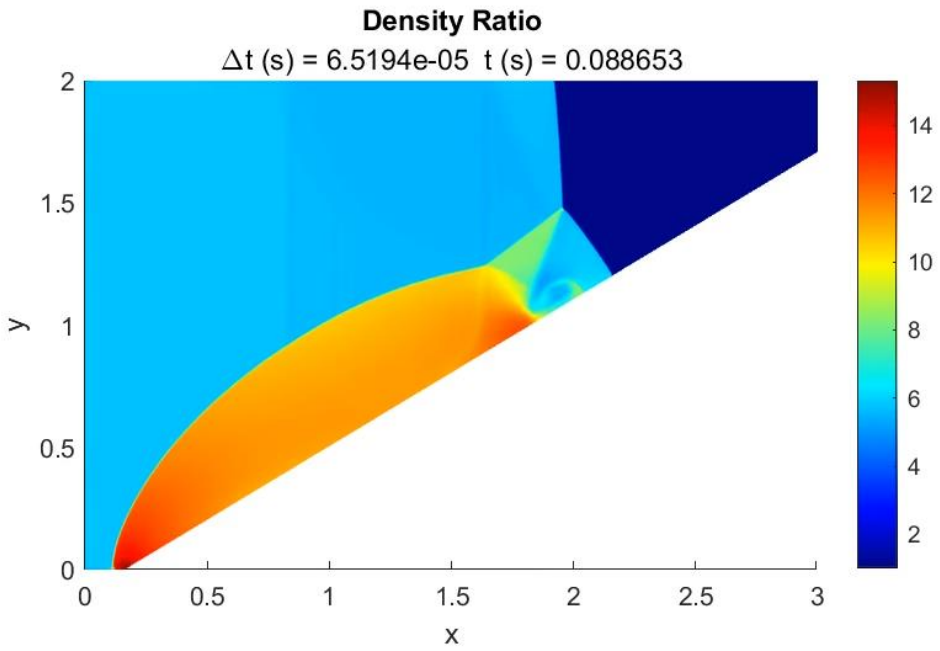
Double Mach Reflection

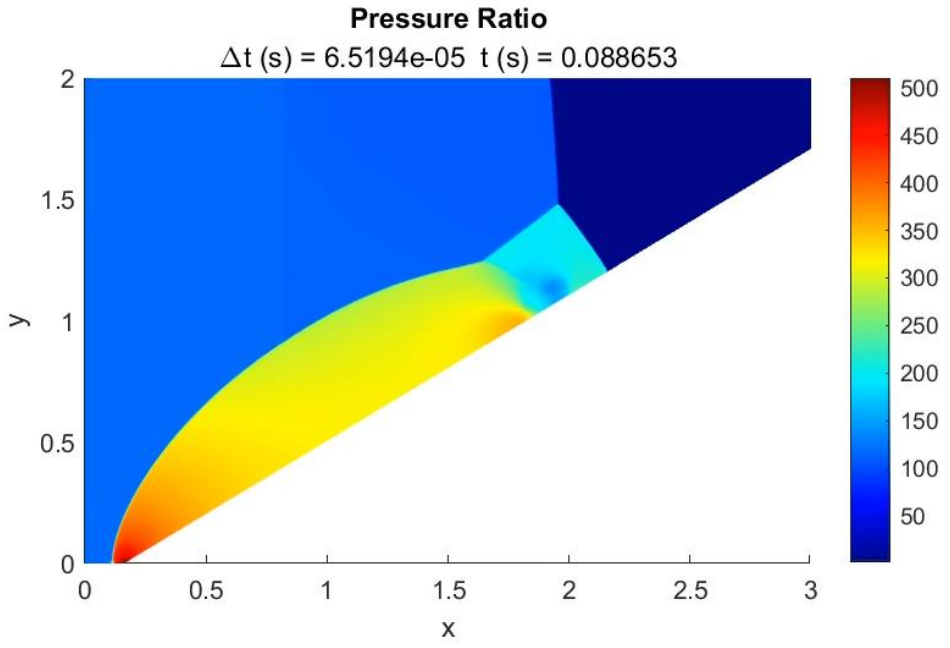
- Test Problem 3
- Grid points x<1/6 m are initialized to give Mach 10 moving shock wave
- dx = 0.003



Double Mach Reflection

- Test Problem 3
- Grid points x<1/6 m are initialized to give Mach 10 moving shock wave
- $\Delta x = 0.003$





Conclusions and Future Work

- Successfully developed a solver for the Euler Equations using the WENO formulation
  - "Prototype code" for future development
  - Many lessons learned about the WENO method while developing the solver
    - WENO polynomials
    - Nonlinear weights
    - Smoothness indicator
    - Riemann solvers and flux splitting
- Obtained good agreement with test cases
- Find use cases in LSP work
- Future Work
  - Create 3D and axisymmetric solver using lessons learned
  - Utilize GPU computing

References

1. Harten, A., Engquist, B., Osher, S., & Chakravarthy, S.R. (1987). Uniformly high order accurate essentially non-oscillatory schemes, 111. *Journal of Computational Physics, 71*, 231-303.
2. Jiang, G., & Shu, C. (1995). Efficient Implementation of Weighted ENO Schemes. *Journal of Computational Physics, 126*, 202-228.
3. Zahran Y. (2017) Weighted non-oscillatory (WENO) methods for solving hyperbolic conservation laws. Lap Lambert Academic Publishing
4. Yao G. (2018) Immersed Boundary Method for CFD: Focusing on its implementation Second Edition
5. Vázquez-Cendón E. (2011) Lecture Notes on Numerical Methods for Hyperbolic Equations: Short Course Book
6. Hesthaven J. (2018) Numerical Methods for Conservation Laws: From Analysis to Algorithms, Society of Industrial and Applied Mathematics